

# Homogenization and uniform resolvent convergence for elliptic operators in a strip perforated along a curve

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## Abstract

We consider an infinite planar straight strip perforated along an infinite curve by small holes located closely one to another. In such domain we consider a general second order elliptic operator subject to classical boundary conditions on the holes. Assuming that the perforation is non-periodic and satisfies rather weak assumptions, we describe possible homogenized problems. Our main result is the proof of the uniform resolvent convergence of the perturbed operator to a homogenized one in various operator norms and the estimates for the rate of convergence. On the basis of the uniform convergence we show the convergence of the spectrum. In a particular case of pure periodic perforation and Dirichlet condition on the boundary of the holes we obtain two-terms asymptotics for the first band functions of the perturbed operator.

## 1 Introduction

Problems in perforated domains is one of the classical objects in the modern homogenization theory. It is impossible to cite all the works in this field and we mention only the books [28], [34], [36], [40], see also the references therein, and some latest papers [14], [15]. One of the possible configurations of the perforation is that along a curve or manifold. It was treated in [2], [19], [20], [21], [26], [27], [31], [32], [33], see also the references therein. The perforation was assumed to be periodic [2], [20], [21], [26], [27], non-periodic [31], [33], [34], or even random [19]. There were considered various geometries of the holes as well as various boundary conditions (both linear and non-linear) on their boundaries. The main result was the classification of the homogenized problems depending on the sizes and distribution of the holes and the convergence of the perturbed solutions to the homogenized ones. The latter was established in a weak or strong sense. Namely, a typical result says that a solution to a perturbed problem converges to a homogenized one weakly or strongly in  $W_2^1$  and strongly in  $L_2$  for each fixed right hand side. In the periodic case under additional symmetries of the holes it is possible to construct complete asymptotic expansions for the perturbed solutions, see [22], [23], [24], [25].

Recently M.Sh. Birman, T.A. Suslina and V.V Zhikov, S.E. Pastukhova initiated a new direction in the homogenization theory. They showed that for the operators with fast periodically oscillating coefficients the uniform resolvent convergence holds, i.e., not only the perturbed solutions converge to the limiting ones, but also the perturbed resolvent converges to the homogenized one in the norm sense, see [3], [4], [41], [42],

[43], [44], [45], [46], [38], [39], [13], and other papers of these authors. The uniform resolvent convergence was established in various operator norms and the estimates for the rates of convergence were obtained. Then a natural question appeared whether similar results can be proven for other types of perturbations in homogenization theory. This question happened to be open for many perturbations usually considered in the homogenization theory, and it is topical. For instance, once the uniform resolvent convergence holds, the convergence of spectrum and spectral projectors is implied by standard theorems in spectral theory [37, Ch. VIII, Sec. 7] and there is no need to prove such convergence independently. The uniform resolvent convergence is also the strongest among all possible ones. The estimates for the rate of uniform convergence is the next step, since they show how close the perturbed resolvent to the homogenized one.

For some periodic perturbations the uniform resolvent convergence was proven in [11], [38], [45] and the estimates for the rate of convergence were obtained (see also the references in the cited works). The results of [45] include the periodic perforation when it covers whole the domain. One more type of perturbations, frequent alternation of boundary conditions was treated in [6], [7], [8], [9]. The uniform resolvent convergence was proven for all possible homogenized problems, for both periodic and non-periodic alternation. The estimates for the rate of the uniform resolvent convergence were obtained. In periodic cases certain asymptotic expansions for the spectra of perturbed operators were constructed.

In the present paper we consider a general second order elliptic operator in an infinite planar strip perforated along an infinite curve. The sizes of the holes and the distance between them are described in terms of two small parameters. The perforation is quite general and no periodicity assumption is made. Namely, both the shapes and the distribution of the holes can be rather arbitrary. On the boundary of the holes we impose one of the classical boundary conditions, i.e., Dirichlet, Neumann, or Robin one. It is allowed to have different boundary conditions on different holes. To the best of authors' knowledge, such mixtures of boundary conditions were not considered before. As the main result we show that depending on the relation between the aforementioned small parameters and boundary conditions on the boundary of the holes the homogenized operator can have Dirichlet boundary condition or delta-interaction on the reference curve along which the perforation is made. In all cases we prove the uniform resolvent convergence of the perturbed operator to the homogenized one and establish the estimates for the rates of convergence. In all cases except one the operator norm is that of the operators from  $L_2$  into  $W_2^1$ , while in the exceptional case it is from  $L_2$  into  $L_2$ . Nevertheless, in the latter case we show that by employing a special boundary corrector one can improve the norm to that for the operators acting from  $L_2$  into  $W_2^1$ . Such kind of results are completely new for the domains perforated periodically along curves or manifolds, while in the present paper we succeeded to study the general periodic perforation with arbitrary boundary conditions. We also consider a particular periodic case with Dirichlet condition on the boundary of the holes. In this case the spectrum has a band structure. Here our result is two-terms asymptotics for the band functions.

Concluding the introduction, we describe briefly the structure of the paper. In the next section we give the precise description of the problem, formulate the main results, and discuss them. In the third section we collect auxiliary lemmata required for the proof of the main result. The forth, fifth, and sixth sections are devoted to the study of the uniform resolvent convergence in various cases. In the last seventh section we obtain asymptotics for the band functions in the periodic case.

## 2 The problem and the main results

Let  $x = (x_1, x_2)$  be Cartesian coordinates in  $\mathbb{R}^2$ ,  $\Omega$  be a horizontal strip of width  $d > 0$ ,  $\Omega := \{x : 0 < x_2 < d\}$ ,  $\gamma$  be an infinite curve lying in  $\Omega$ , separated from  $\partial\Omega$  by a fixed distance, and having an uniformly bounded curvature. We assume that the curve  $\gamma$  is

$C^2$ -smooth, has no self-intersection,  $s$  is its arc length,  $s \in (-\infty, +\infty)$ , and  $\rho = \rho(s)$  is the vector function describing the curve  $\gamma$ . We also suppose that the distance between two points on the curve  $\gamma$  increases as the length of the curve between these points does, namely, there exists a positive constant  $c_6$  such that for all  $s, \tilde{s} \in \mathbb{R}$

$$|\rho(s) - \rho(\tilde{s})| \geq c_6 |s - \tilde{s}|.$$

Since the curvature of  $\gamma$  is uniformly bounded and the curve  $\gamma$  is infinite, it splits the domain  $\Omega$  into two disjoint subdomains. The upper one is denoted by  $\Omega_+$  and the lower one is  $\Omega_-$ . By  $B_r(a)$  we denote the ball in  $\mathbb{R}^2$  of radius  $r$  centered at  $a$ .

Let  $s_k \in \mathbb{R}$ ,  $k \in \mathbb{Z}$ , be a strictly monotonically increasing sequence satisfying the inequality

$$0 < c_0 \leq s_{k+1} - s_k \leq c_0^{-1}, \quad k \in \mathbb{Z}, \quad (2.1)$$

with a constant  $c_0$  independent of  $j$ . By  $\omega_k$ ,  $k \in \mathbb{Z}$ , we indicate a sequence of bounded domains in  $\mathbb{R}^2$  having  $C^2$ -boundaries. Denoting by  $\varepsilon$  a small positive parameter, we introduce domains

$$\begin{aligned} \theta^\varepsilon &:= \theta_0^\varepsilon \cup \theta_1^\varepsilon, \quad \theta_i^\varepsilon := \bigcup_{k \in \mathbb{M}_i} \omega_k^\varepsilon, \quad i = 0, 1, \\ \omega_k^\varepsilon &:= \{x : \varepsilon^{-1} \eta^{-1}(\varepsilon)(x - y_k^\varepsilon) \in \omega_k\}, \quad y_k^\varepsilon := \rho(s_k \varepsilon), \end{aligned}$$

where

$$\mathbb{M}_i \subset \mathbb{Z}, \quad \mathbb{M}_0 \cap \mathbb{M}_1 = \emptyset, \quad \mathbb{M}_0 \cup \mathbb{M}_1 = \mathbb{Z},$$

and  $\eta = \eta(\varepsilon)$  is a some function and  $0 < \eta(\varepsilon) \leq 1$ .

We make the following assumptions.

(A1). There exists a fixed number  $R_1$  such that for each  $k \in \mathbb{Z}$  there exists a point  $x^k \in \mathbb{R}^2$  such that  $B_{R_1}(x^k) \subset \omega_k$ .

(A2). There exists a fixed ball  $B_{R_2}(0)$  and a number  $b > 1$  such that

$$\begin{aligned} \omega_k &\subset B_{R_2}(0) \quad \text{for all } k \in \mathbb{Z}, \\ B_{bR_2\varepsilon}(y_k^\varepsilon) \cap B_{bR_2\varepsilon}(y_i^\varepsilon) &= \emptyset \quad \text{for all } i \neq k, \quad i, k \in \mathbb{Z}, \end{aligned}$$

and for all sufficiently small  $\varepsilon$ .

(A3). For each  $\delta > 0$  and all  $u \in W_2^1(B_{bR_2}(0) \setminus \omega_k)$ ,  $k \in \mathbb{Z}$ , the inequality

$$\|u\|_{L_2(\partial\omega_k)}^2 \leq \delta \|\nabla u\|_{L_2(B_{bR_2}(0) \setminus \omega_k)}^2 + c_1(\delta) \|u\|_{L_2(B_{bR_2}(0) \setminus B_{R_2}(0))}^2$$

holds true, where  $c_1(\delta)$  is a positive constant independent of  $u$  and  $j$ .

By  $A_{ij} = A_{ij}(x)$ ,  $A_i = A_i(x)$ ,  $A_0 = A_0(x)$  we denote some functions satisfying the conditions

$$\begin{aligned} A_{ij}, A_i &\in W_\infty^1(\Omega), \quad i, j = 1, 2, \quad A_0 \in L_\infty(\Omega), \\ A_{ji} &= A_{ji}, \quad A_{ij}, A_0 \text{ are real}, \\ \sum_{i,j=1}^2 A_{ij} \xi_i \xi_j &\geq c_2 |\xi|^2, \quad x \in \Omega, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \end{aligned} \quad (2.2)$$

where  $c_2$  is a positive constant independent of  $x$  and  $\xi$ .

In a small neighborhood of  $\gamma$  we introduce local coordinates  $(s, \tau)$ , where  $\tau$  is the distance to a point measured along the normal  $\nu$ , and  $s$ , we remind, is the arc length of  $\gamma$ . Since the curvature of  $\gamma$  is uniformly bounded, the coordinates  $(s, \tau)$  are well-defined for  $|\tau| < \tau_0$ ,  $s \in \mathbb{R}$ , where  $\tau_0$  is a sufficiently small fixed positive number.

In this paper we study a singularly perturbed operator depending on  $\varepsilon$  which we denote as  $\mathcal{H}^\varepsilon$ . It is introduced by the differential expression

$$- \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^2 A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 \quad (2.3)$$

in  $\Omega^\varepsilon := \Omega \setminus \theta^\varepsilon$  subject to the Dirichlet condition on  $\partial\Omega \cup \partial\theta_0^\varepsilon$  and to the Robin condition

$$\left(\frac{\partial}{\partial N^\varepsilon} + a\right)u = 0 \quad \text{on} \quad \partial\theta_1^\varepsilon, \quad \frac{\partial}{\partial N^\varepsilon} := \sum_{i,j=1}^2 A_{ij}\nu_i^\varepsilon \frac{\partial}{\partial x_j} + \sum_{j=1}^2 \bar{A}_j\nu_j^\varepsilon,$$

where  $\nu^\varepsilon = (\nu_1^\varepsilon, \nu_2^\varepsilon)$  is the inward normal to  $\theta_1^\varepsilon$ ,  $a = a(x)$  is a some function defined for  $|\tau| < \tau_0$  and  $a \in W_\infty^1(\{x : |\tau| < \tau_0\})$ .

Rigorously we define the operator  $\mathcal{H}^\varepsilon$  as the lower-semibounded self-adjoint operator in  $L_2(\Omega^\varepsilon)$  associated with the closed lower-semibounded symmetric sesquilinear form

$$\begin{aligned} \mathfrak{h}^\varepsilon(u, v) := & \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega^\varepsilon)} + \sum_{j=1}^2 \left( A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega^\varepsilon)} \\ & + \sum_{j=1}^2 \left( u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega^\varepsilon)} + (A_0 u, v)_{L_2(\Omega^\varepsilon)} + (au, v)_{L_2(\partial\theta_1^\varepsilon)} \end{aligned}$$

in  $L_2(\Omega^\varepsilon)$  on  $\dot{W}_2^1(\Omega^\varepsilon, \partial\Omega \cup \partial\theta_0^\varepsilon)$ . Hereinafter for any domain  $Q \subset \mathbb{R}^2$  and any curve  $S \subset Q$  by  $\dot{W}_2^1(Q, S)$  we denote the subspace of  $W_2^1(Q)$  consisting of the functions having zero trace on  $S$ , and we let  $\dot{W}_2^1(Q) := \dot{W}_2^1(Q, \partial Q)$ . If else is not said, in what follows all the differential operators are introduced in this way, i.e., they will be self-adjoint lower semibounded operators in  $L_2(\Omega)$  or  $L_2(\Omega^\varepsilon)$  associated with closed lower-semibounded symmetric sesquilinear form. Below for the sake of brevity we shall just write the differential expression with the boundary condition as well as the associated form.

Our main aim is to study the resolvent convergence and the spectrum's behavior of the operator  $\mathcal{H}^\varepsilon$ . To formulate our main results, we need additional notations.

By  $\mathcal{H}_D^0$  we denote the operator in  $L_2(\Omega)$  with the differential expression (2.3) subject to the Dirichlet condition on  $\gamma$  and  $\partial\Omega$ . The associated form is

$$\begin{aligned} \mathfrak{h}_D^0(u, v) := & \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega)} + \sum_{j=1}^2 \left( A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega)} \\ & + \sum_{j=1}^2 \left( u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega)} + (A_0 u, v)_{L_2(\Omega)} \end{aligned} \quad (2.4)$$

in  $L_2(\Omega)$  on  $\dot{W}_2^1(\Omega, \partial\Omega \cup \gamma)$ . In the same way we introduce the operator  $\mathcal{H}^0$  with the differential expression (2.3) subject to the Dirichlet condition on  $\partial\Omega$  and with no condition on  $\gamma$ . The corresponding form  $\mathfrak{h}^0$  is again given by (2.4), but on the domain  $\dot{W}_2^1(\Omega)$ . By analogy with [5, Lem. 2.2], [35, Ch. IV, Sec. 2.2, 2.3], [12, Lem. 3.2] one can check that the domains of the operators  $\mathcal{H}_D^0$ ,  $\mathcal{H}^0$  are given by the identities

$$\mathfrak{D}(\mathcal{H}_D^0) = \dot{W}_2^1(\Omega, \partial\Omega \cup \gamma) \cap W_2^2(\Omega \setminus \gamma), \quad \mathfrak{D}(\mathcal{H}^0) = \dot{W}_2^1(\Omega) \cap W_2^2(\Omega).$$

By  $i$  we denote the imaginary unit. We shall employ the symbol  $\|\cdot\|_{X \rightarrow Y}$  to indicate the norm of an operator acting from a Banach space  $X$  to a Banach space  $Y$ .

Now we are ready to formulate our first main result.

**Theorem 2.1.** *Suppose (A1), (A2), (A3), and*

(A4). *The convergence  $\varepsilon \ln \eta(\varepsilon) \rightarrow 0$ ,  $\varepsilon \rightarrow +0$  holds true.*

(A5). *There exists a constant  $R_3$  such that*

$$\theta^\varepsilon \subset \bigcup_{k \in \mathbb{M}_0} B_{R_3\varepsilon}(y_k^\varepsilon), \quad \omega_k^\varepsilon \subset B_{R_3\varepsilon}(y_k^\varepsilon), \quad k \in \mathbb{M}_0.$$

(A6). For  $R_2$  from (A2) and all  $k \in \mathbb{M}_1$ ,  $u \in \dot{W}_2^1(B_{bR_2}(0) \setminus \omega_k, \partial B_{bR_2}(0))$  the inequality

$$\|u\|_{L_2(B_{bR_2}(0) \setminus \omega_k)} \leq C \|\nabla u\|_{L_2(B_{bR_2}(0) \setminus \omega_k)}$$

holds true, where  $C$  is a positive constant independent of  $j$  and  $u$ .

Then the estimate

$$\|(\mathcal{H}^\varepsilon - i)^{-1} - (\mathcal{H}_D^0 - i)^{-1}\|_{L_2(\Omega) \rightarrow W_2^1(\Omega^\varepsilon)} \leq C\varepsilon^{1/2}(|\ln \eta(\varepsilon)|^{1/2} + 1)$$

holds true, where  $C$  is a positive constant independent of  $\varepsilon$ .

Let  $\nu^0 = (\nu_1^0, \nu_2^0)$  be the normal to  $\gamma$  which is inward for  $\Omega_-$ , and

$$\frac{\partial}{\partial N^0} := \sum_{i,j=1}^2 A_{ij} \nu_i^0 \frac{\partial}{\partial x_j}.$$

By  $[\cdot]_\gamma$  we indicate the jump of a function on  $\gamma$ ,

$$[u]_\gamma = u|_{\tau=+0} - u|_{\tau=-0}. \quad (2.5)$$

Given a function  $\beta = \beta(s)$  in  $W_\infty^1(\gamma)$ , we introduce the operator  $\mathcal{H}_\beta^0$  with the differential expression (2.3) subject to the boundary conditions

$$[u]_\gamma = 0, \quad \left[ \frac{\partial u}{\partial N^0} \right]_\gamma + \beta u|_\gamma = 0. \quad (2.6)$$

The associated form is

$$\begin{aligned} \mathfrak{h}_\beta^0(u, v) := & \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega)} + \sum_{j=1}^2 \left( A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega)} \\ & + \sum_{j=1}^2 \left( u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega)} + (A_0 u, v)_{L_2(\Omega)} + (\beta u, v)_{L_2(\gamma)} \end{aligned}$$

in  $L_2(\Omega)$  on  $\dot{W}_2^1(\Omega)$ . Again by analogy with [5, Lem. 2.2], [35, Ch. IV, Sec. 2.2, 2.3], [12, Lem. 3.2] one can show that

$$\mathfrak{D}(\mathcal{H}_\beta^0) = \{u \in \dot{W}_2^1(\Omega) : u \in W_2^2(\Omega_\pm) \text{ and (2.6) is satisfied}\}.$$

If  $\beta = 0$ , instead of  $\mathcal{H}_0^0$  we shall simply write  $\mathcal{H}^0$ . As one can see, in this case there is no jump of the normal derivative in (2.6) and the domain of  $\mathcal{H}^0$  is just  $\dot{W}_2^2(\Omega)$ .

In the next theorem we deal with the case when the perturbed operator involves the Dirichlet condition at least on a part of  $\partial\theta^\varepsilon$  but in contrast to (A4), the function  $\varepsilon \ln \eta(\varepsilon)$  converges either to a non-zero constant or to infinity.

**Theorem 2.2.** Suppose (A1), (A2), (A3), (A5), and

(A12). The convergence  $(\varepsilon \ln \eta(\varepsilon))^{-1} \rightarrow -K$ ,  $\varepsilon \rightarrow +0$ , holds true, where  $K$  is a non-negative constant.

(A13). The set  $\mathbb{M}_0$  is non-empty.

(A14). For  $b$  and  $R_2$  from (A2) let

$$\alpha^\varepsilon(s) := \begin{cases} \frac{\pi}{bR_2}, & |s - \varepsilon s_k| < bR_2\varepsilon\eta, \quad k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

There exists a function  $\alpha = \alpha(s)$  in  $W_\infty^1(\gamma)$  and a function  $\varkappa = \varkappa(\varepsilon)$ ,  $\varkappa(\varepsilon) \rightarrow +0$ ,  $\varepsilon \rightarrow +0$  such that for all  $n \in \mathbb{Z}$  and all sufficiently small  $\varepsilon$  the estimates (2.11) are valid.

Denote

$$\beta := -\alpha \frac{(K + \mu)}{A_{11}A_{22} - A_{12}^2}, \quad \beta_0 := -\alpha \frac{K}{A_{11}A_{22} - A_{12}^2}, \quad \mu(\varepsilon) := -\frac{1}{\varepsilon \ln \eta(\varepsilon)} - K.$$

Then the estimates

$$\|(\mathcal{H}^\varepsilon - \mathbf{i})^{-1}f - (\mathcal{H}_\beta^0 - \mathbf{i})^{-1}f\|_{L_2(\Omega) \rightarrow L_2(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon)) \quad (2.7)$$

$$\|(\mathcal{H}^\varepsilon - \mathbf{i})^{-1} - (\mathcal{H}_{\beta_0}^0 - \mathbf{i})^{-1}\|_{L_2(\Omega) \rightarrow L_2(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon) + \mu(\varepsilon)) \quad (2.8)$$

hold true, where  $C$  is a positive constant independent of  $\varepsilon$ . There exists an explicit function  $W^\varepsilon$  defined in (6.6) such that the estimate

$$\|(\mathcal{H}^\varepsilon - \mathbf{i})^{-1} - W^\varepsilon(\mathcal{H}_\beta^0 - \mathbf{i})^{-1}\|_{L_2(\Omega) \rightarrow W_2^1(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon)) \quad (2.9)$$

is valid, where  $C$  is a positive constant independent of  $\varepsilon$ . If  $K = 0$ , the estimate

$$\|(\mathcal{H}^\varepsilon - \mathbf{i})^{-1} - (\mathcal{H}_{\beta_0}^0 - \mathbf{i})^{-1}\|_{L_2(\Omega) \rightarrow W_2^1(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \mu^{1/2}(\varepsilon)) \quad (2.10)$$

holds true, where  $C$  is a positive constant independent of  $\varepsilon$ .

The next two theorems concern the case when  $\mathbf{M}_0$  is empty, i.e., the perturbed operator involves just the Robin condition on  $\partial\theta^\varepsilon$ .

**Theorem 2.3.** Suppose (A1), (A2), (A3), and

(A7). The convergence  $\eta(\varepsilon) \rightarrow 0$ ,  $\varepsilon \rightarrow +0$  holds true.

(A8). The set  $\mathbf{M}_0$  is empty.

Then the estimates

$$\|(\mathcal{H}^\varepsilon - \mathbf{i})^{-1} - (\mathcal{H}^0 - \mathbf{i})^{-1}\|_{L_2(\Omega) \rightarrow W_2^1(\Omega^\varepsilon)} \leq C\eta(\varepsilon)(|\ln \eta(\varepsilon)| + 1),$$

if  $a \not\equiv 0$ , and

$$\|(\mathcal{H}^\varepsilon - \mathbf{i})^{-1}f - (\mathcal{H}^0 - \mathbf{i})^{-1}f\|_{L_2(\Omega) \rightarrow W_2^1(\Omega^\varepsilon)} \leq C\varepsilon^{1/2}\eta(\varepsilon)(|\ln \eta(\varepsilon)|^{1/2} + 1),$$

if  $a \equiv 0$ , hold true, where  $C$  is a positive constant independent of  $\varepsilon$ .

**Theorem 2.4.** Suppose (A1), (A2), (A3), (A8), and

(A9). The identity  $\eta = \text{const}$  holds true.

(A10). For  $b$  and  $R_2$  from (A2) let

$$\alpha^\varepsilon(s) := \begin{cases} \frac{\eta|\partial\omega_k|}{2bR_2}, & |s - \varepsilon s_k| < bR_2\varepsilon\eta, \quad k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

There exists a function  $\alpha = \alpha(s)$  in  $W_\infty^1(\gamma)$  and a function  $\varkappa = \varkappa(\varepsilon)$ ,  $\varkappa(\varepsilon) \rightarrow +0$ ,  $\varepsilon \rightarrow +0$  such that for all  $n \in \mathbb{Z}$  and all sufficiently small  $\varepsilon$  the estimates

$$\int_n^{n+1} |\alpha^\varepsilon(s) - \alpha(s)| ds \leq \varkappa(\varepsilon) \quad (2.11)$$

are valid.

(A11). For  $b$  and  $R_2$  from (A2) and  $k \in \mathbb{Z}$  there exists a solution to the boundary value problem

$$\begin{aligned} \Delta X_k &= 0 \quad \text{in } B_{\frac{b+1}{2}R_2}(0) \setminus \omega_k, \\ \frac{\partial X_k}{\partial \nu} &= 1 \quad \text{on } \partial \omega_k, \quad \frac{\partial X_k}{\partial \nu} = -\frac{|\partial \omega_k|}{\pi(b+1)R_2} \quad \text{on } \partial B_{\frac{b+1}{2}R_2}(0), \\ &\text{belonging to } C^1(\overline{B_{\frac{b+1}{2}R_2}(0) \setminus \omega_k}) \text{ and satisfying the uniform in } k \in \mathbb{Z} \text{ estimate} \end{aligned}$$

$$\|X_k\|_{C^1(\overline{B_{\frac{b+1}{2}R_2}(0) \setminus \omega_k})} \leq C. \quad (2.12)$$

Here  $\nu$  is the outward normal to  $\partial(B_{\frac{b+1}{2}R_2}(0) \setminus \omega_k)$ .

Then the estimate

$$\|(\mathcal{H}^\varepsilon - i)^{-1} - (\mathcal{H}_{\alpha a}^0 - i)^{-1}\|_{L_2(\Omega) \rightarrow W_2^1(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon))$$

holds true, where  $C$  is a positive constant independent of  $\varepsilon$ .

Let us discuss the main results. General assumption (A1) just says that there exist two fixed balls such that all the domains  $\omega_j$  can be inscribed in the first ball and the second ball can be inscribed in all  $\omega_j$ . Assumption (A2) and (2.1) control the distance between the holes. Assumption (A3) is just a restriction for the geometry of the boundaries  $\partial \omega_k$  saying they should behave quite regularly for large  $k$  to satisfy the required estimate. The latter estimate obviously holds for each fixed  $k$  but with a constant depending on  $k$ ; assumption (A3) restricts to the case when this constant can be chosen independent of  $k$ .

According to Theorem 2.1, if the sizes of the holes are not too small (assumption (A4)) and the holes with the Dirichlet condition are, roughly speaking, distributed “uniformly” (assumption (A5)), the homogenized operator is subject to the Dirichlet condition on  $\gamma$  and we have the uniform resolvent condition in the sense of the operator norm  $\|\cdot\|_{L_2(\Omega) \rightarrow L_2(\Omega^\varepsilon)}$ . As one can see, assumption (A4) means that the sizes of the holes can be much smaller than the distances between them (for instance,  $\eta(\varepsilon) = \varepsilon^\alpha$ ,  $\alpha = \text{const} > 0$ ) but nevertheless the homogenized operator is still subject to the Dirichlet condition on  $\gamma$ . This phenomenon is close to a similar one in the problems with frequent alternation of boundary conditions, see, for instance, [9], [18]. Assumption (A6) is again the restriction for the geometries of  $\omega_k$  for large  $k$ .

If the function  $\varepsilon \ln \eta(\varepsilon)$  goes to a constant or to infinity as  $\varepsilon \rightarrow +0$ , according to Theorem 2.2, the homogenized operator has boundary condition (2.6) instead of the Dirichlet one. This boundary condition describes a delta-interaction on  $\gamma$ , see, for instance, [1, App. K, Sec. K.4.1]. The uniform resolvent convergence holds in the sense of the operator norm  $\|\cdot\|_{L_2(\Omega) \rightarrow L_2(\Omega^\varepsilon)}$  only. To improve the norm, one has either to employ the boundary corrector, see (2.9), or to assume additionally  $K = 0$ , see (2.10). We observe that according to assumption (A14), the coefficient  $\beta$  in (2.6) for the homogenized operator depends only on the distribution of the points  $s_k$  and there is no dependence on the geometries of the holes. There is also no special restrictions for the part  $\partial \theta_0^\varepsilon$  with the Dirichlet condition. For instance, the number of holes in  $\partial \theta_0^\varepsilon$  can be finite or infinite and the structure of this set can be rather arbitrary. We mention that similar situation holds for the problems with frequent alternation of boundary conditions with the Dirichlet conditions on exponentially small parts of the boundary, cf. [9].

If the perturbed operator has no Dirichlet condition on  $\partial \theta^\varepsilon$ , i.e., if  $\mathbf{M}_0$  is empty, the homogenized operator has either condition (2.6) on  $\gamma$  (Theorem 2.4) or even no condition (Theorem 2.3). In both cases we again have the uniform resolvent convergence in the operator norm  $\|\cdot\|_{L_2(\Omega) \rightarrow W_2^1(\Omega^\varepsilon)}$ . In Theorem 2.3 we need no additional restrictions for the geometries of  $\partial \omega_k$ , this is owing to assumption (A7). In Theorem 2.4  $\eta$  is



constant (see (A8)) and because of this we introduce two additional assumptions for  $\partial\omega_k$ . Assumption (A10) restricts the distribution of lengths of  $\partial\omega_k$ , see (2.12), while assumption (A11) describes the “regularity” of  $\partial\omega_k$  as  $|k| \rightarrow \infty$  like (A6) in Theorem 2.1. We stress that the coefficient  $\beta$  in (2.6) for the homogenized operator depends both on the distribution of the holes and the sizes of their boundaries.

Our next main result describes the convergence of the spectrum of  $\mathcal{H}^\varepsilon$  as  $\varepsilon \rightarrow +0$ .

**Theorem 2.5.** *Under the hypotheses of Theorems 2.1, 2.3, 2.4, 2.2 the spectrum of the perturbed operator  $\mathcal{H}^\varepsilon$  converges to that of the corresponding homogenized operator. Namely, if  $\lambda$  is not in the spectrum of the homogenized operator, for sufficiently small  $\varepsilon$  the same is true for the perturbed operator. And if  $\lambda$  is in the spectrum of the homogenized operator, there exists  $\lambda_\varepsilon$  in the spectrum of the perturbed operator such that  $\lambda_\varepsilon \rightarrow \lambda$  as  $\varepsilon \rightarrow +0$ .*

We note that this theorem is not directly implied by Theorems 2.1, 2.3, 2.4, 2.2. Despite these theorems state convergence of the perturbed resolvent to the homogenized one in the uniform norm sense, the norm itself depends on  $\varepsilon$ . Nevertheless, this makes no serious troubles and in the proof of Theorem 2.5 we show a simple extension of the perturbed operator to have the classical uniform resolvent convergence.

In the cases of general non-periodic perforation treated in Theorems 2.1, 2.3, 2.4, 2.2 the structure of the spectrum of  $\mathcal{H}^\varepsilon$  can be very complicated and in fact we can only describe its convergence by Theorem 2.5. If we pose additional assumptions for the perforation and the operator, it is possible to get more information about the spectrum’s behavior. The most obvious example is the periodic perforation for the Laplace operator. Exactly this case is studied in the second part of the paper.

We assume that  $A_{ij} = \delta_{ij}$ ,  $A_j = 0$ ,  $A_0 = 0$ , i.e.,  $\mathcal{H}^\varepsilon = -\Delta$ . The curve  $\gamma$  is supposed to be a straight horizontal line  $\gamma = \{x : x_2 = d_0\}$ ,  $d_0 \in (0, d)$ , and  $s_k = \pi k$ . All the domains  $\omega_j$  are supposed to be the same,  $\omega_j = \omega$ ,  $j \in \mathbb{Z}$ . We shall consider only the case as  $\mathbb{M}_1$  is empty. If this set is non-empty, the situation becomes rather complicated and this is subject to a separate paper.

Since now the perforation is periodic, assumptions (A1), (A2), (A3), (A5), (A6), (A14) become trivial and obviously hold true.

Under the above restrictions the operator  $\mathcal{H}^\varepsilon$  is periodic and its spectrum is described by Floquet-Bloch theory (see, for instance, [29]). Namely, we introduce the periodicity cell  $\square^\varepsilon := \{x : |x_1| < \varepsilon\pi/2, 0 < x_2 < d\}$  and consider the operator  $\mathcal{H}^\varepsilon(\tau)$

$$\mathcal{H}^\varepsilon(\tau) := \left( i \frac{\partial}{\partial x_1} - \frac{\varsigma}{\varepsilon} \right)^2 - \frac{\partial^2}{\partial x_2^2} \quad \text{in } \square^\varepsilon \setminus \omega^\varepsilon, \quad \varsigma \in [-1, 1],$$

$$\omega^\varepsilon := \{x : \varepsilon^{-1}\eta^{-1}(\varepsilon)(x_1, x_2 - d_0) \in \omega\},$$

subject to Dirichlet condition on  $\Gamma^\varepsilon := \partial\Omega \cap \partial\square^\varepsilon$  and on  $\partial\omega^\varepsilon$  and to periodic boundary condition on  $\partial\square^\varepsilon \setminus \overline{\Gamma}^\varepsilon$ . The associated quadratic form is

$$\mathfrak{h}_\tau^\varepsilon(u, v) := \left( \left( i \frac{\partial}{\partial x_1} - \frac{\varsigma}{\varepsilon} \right) u, \left( i \frac{\partial}{\partial x_1} - \frac{\varsigma}{\varepsilon} \right) v \right)_{L_2(\square^\varepsilon \setminus \omega^\varepsilon)} + \left( \frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2} \right)_{L_2(\square^\varepsilon \setminus \omega^\varepsilon)} \quad (2.13)$$

on  $\dot{W}_{2,per}^1(\square^\varepsilon \setminus \omega^\varepsilon, \partial\omega^\varepsilon \cup \Gamma^\varepsilon)$ . Here  $\dot{W}_{2,per}^1(\square^\varepsilon \setminus \omega^\varepsilon, S)$  is the subspace of the functions in  $\dot{W}_2^1(\square \setminus \omega^\varepsilon, S)$  satisfying periodic boundary condition on  $\partial\square^\varepsilon \setminus \overline{\Gamma}^\varepsilon$ .

By  $\lambda_n(\varsigma, \varepsilon)$ ,  $n \geq 1$ , we denote the eigenvalues of the operator  $\mathcal{H}^\varepsilon(\tau)$  taken in the ascending order counting multiplicities. Then

$$\sigma(\mathcal{H}^\varepsilon) = \bigcup_{n=1}^{\infty} \{\lambda_n(\varsigma, \varepsilon) : \varsigma \in [-1, 1]\}, \quad (2.14)$$

where  $\sigma(\cdot)$  denotes the spectrum of an operator.

We proceed to the limiting operators. By  $\mathfrak{L}^\varepsilon$  we denote the subspace of  $L_2(\square^\varepsilon)$  consisting of the functions independent of  $x_1$ . We decompose  $L_2(\square^\varepsilon)$  as  $L_2(\square^\varepsilon) =$



$\mathfrak{L}^\varepsilon \oplus \mathfrak{L}_\perp^\varepsilon$ , where  $\mathfrak{L}_\perp^\varepsilon$  is the orthogonal complement to  $\mathfrak{L}^\varepsilon$  in  $L_2(\square^\varepsilon)$ . In  $\mathfrak{L}^\varepsilon$  we introduce the operator  $\mathcal{Q}_D^0 := -\frac{d^2}{dx_2^2}$  subject to Dirichlet condition at  $x_2 = 0$ ,  $x_2 = d_0$ ,  $x_2 =$

$d$ . The associated sesquilinear form is  $\left(\frac{du}{dx_2}, \frac{dv}{dx_2}\right)_{L_2(0,d)}$  on  $\dot{W}_2^1((0,d), \{0, d_0, d\})$ . The eigenvalues of  $\mathcal{Q}_D^0$  are  $\pi^2/(m^2 d_0^2)$ ,  $\pi^2/(k^2(d-d_0)^2)$ ,  $m, k \in \mathbb{N}$ . The first sequence corresponds to the interval  $(0, d_0)$ , while the other to  $(d_0, d)$ . We take these eigenvalues in the ascending order counting multiplicities denoting then by  $\Lambda_n^D$ ,  $n \geq 1$ . We observe that  $\Lambda_n^D$  can be a double eigenvalue if  $\Lambda_n^D = \pi^2/(m^2 d_0^2) = \pi^2/(k^2(d-d_0)^2)$  for some  $m, k$ .

**Theorem 2.6.** *Suppose (A4) and*

(A15). *The set  $\mathbb{M}_1$  is empty.*

(A16). *The parameter  $\varsigma$  ranges in the segment  $|\varsigma| < 1 - \varsigma_0$ , where  $\varsigma_0 \in (0, 1)$ .*

*Then for sufficiently small  $\varepsilon$  the estimate*

$$\left\| \left( \mathcal{H}^\varepsilon(\tau) - \frac{\tau^2}{\varepsilon^2} \right)^{-1} - (\mathcal{Q}_D^0)^{-1} \oplus 0 \right\|_{L_2(\square^\varepsilon) \rightarrow L_2(\square^\varepsilon \setminus \omega^\varepsilon)} \leq C_0 \varepsilon^{1/2} (|\ln \eta|^{1/2} + 1) \quad (2.15)$$

*holds true, where  $C_0$  is a constant independent of  $\varepsilon$ . Given  $N \in \mathbb{N}$ , there exists  $\varepsilon_0 = \varepsilon_0(N)$  such that the eigenvalues  $\lambda_n(\tau, \varepsilon)$ ,  $n = 1, \dots, N$ , satisfy the relations*

$$\left| \lambda_n(\tau, \varepsilon) - \frac{\tau^2}{\varepsilon^2} - \Lambda_n^D \right| \leq 4C_0 (\Lambda_n^D)^2 \varepsilon^{1/2} (|\ln \eta|^{1/2} + 1) \quad (2.16)$$

*for  $\varepsilon < \varepsilon_0$ . The identity*

$$\inf \sigma(\mathcal{H}^\varepsilon) = \lambda_1(0, \varepsilon) \quad (2.17)$$

*holds true.*

In  $\mathfrak{L}^\varepsilon$  we introduce the operator  $\mathcal{Q}_K^0(\mu) := -\frac{d^2}{dx_2^2}$  subject to Dirichlet condition at  $x_1 = 0$ ,  $x_1 = d$ , and to the condition

$$[u]_{x_2=d_0} = 0, \quad \left[ \frac{du}{dx_2} \right]_{x_2=d_0} - 2(K + \mu)u|_{x_2=d_0} = 0,$$

where  $[\cdot]_{x_2=d_0}$  denotes the jump at  $x_2 = 0$  like in (2.5). The associated sesquilinear form is  $\left(\frac{du}{dx_2}, \frac{dv}{dx_2}\right)_{L_2(0,d)} + 2(K + \mu)u(0)\overline{v(0)}$  on  $\dot{W}_2^1((0,d), \{0, d\})$ . The eigenvalues of the operator  $\mathcal{Q}_K^0(\mu)$  are the roots to the equation

$$\sqrt{\Lambda} \sin \sqrt{\Lambda} d + \sin \sqrt{\Lambda} d_0 \sin \sqrt{\Lambda} (d - d_0) = 0.$$

We take them in ascending order counting multiplicities and denote then by  $\Lambda_n^K(\mu)$ ,  $n \geq 1$ .

**Theorem 2.7.** *Suppose (A12), (A15), (A16). Then for sufficiently small  $\varepsilon$  the estimate*

$$\left\| \left( \mathcal{H}^\varepsilon(\tau) - \frac{\tau^2}{\varepsilon^2} \right)^{-1} - (\mathcal{Q}_K^0(\mu))^{-1} \oplus 0 \right\|_{L_2(\square^\varepsilon) \rightarrow L_2(\square^\varepsilon \setminus \omega^\varepsilon)} \leq C_0 \varepsilon^{1/2} \quad (2.18)$$

*holds true, where  $C_0$  is a constant independent of  $\varepsilon$ . Given  $N \in \mathbb{N}$ , there exists  $\varepsilon_0 = \varepsilon_0(N)$  such that the eigenvalues  $\lambda_n(\tau, \varepsilon)$ ,  $n = 1, \dots, N$ , satisfy the relations*

$$\left| \lambda_n(\tau, \varepsilon) - \frac{\tau^2}{\varepsilon^2} - \Lambda_n^K(\mu) \right| \leq 4C_0 (\Lambda_n^K(\mu))^2 \varepsilon^{1/2} \quad (2.19)$$

*for  $\varepsilon < \varepsilon_0$ . The identity (2.17) holds true.*

Two latter theorems provide the asymptotics for the first band functions of the perturbed operator in the considered periodic case as well as the uniform resolvent convergence for the cell operator  $\mathcal{H}^\varepsilon(\tau)$ . Estimates (2.15), (2.18) show that after an appropriate spectral shift the resolvent of the cell operator behaves as a one-dimensional operator; this is reflected in the asymptotics for the first band functions in (2.16), (2.19). The eigenvalues  $\Lambda_n^K(\mu)$  are holomorphic w.r.t.  $\mu$  as one can see easily from their definition. The bottom of the spectrum of the perturbed operator is attained by the first band function at  $\tau = 0$ , see (2.17).

Throughout the rest of the paper by  $C, C_1, C_2, C_3, \dots$  we shall indicate various inessential positive constants independent of  $\varepsilon, \eta(\varepsilon), s, \tau, x$ , and various functions  $f, u, v, \dots$  from the Sobolev spaces we shall deal with. In all the estimates such constants are independent on the functions written explicitly (except the coefficients  $A_{ij}, A_j, A_0, a$ ). In the case of local estimates in a vicinity of each  $\omega_k^\varepsilon$  such constants are also supposed to be independent of  $k$ .

### 3 Preliminaries

In this section we prove several auxiliary lemmata which will be employed in the proof of our main results in the subsequent sections. In all the lemmata we assume (A1), (A2), (A3).

**Lemma 3.1.** *The perimeters and the areas of  $\omega_k$  satisfy the uniform in  $k$  estimates*

$$C_1 \leq |\partial\omega_k| \leq C_2, \quad C_3 \leq |\omega_k| \leq C_4.$$

*Proof.* Both the lower bounds follow directly from assumption (A1) with  $C_1 = 2\pi R_1$ ,  $C_3 = \pi R_1^2$ . The upper bound for the perimeters is implied by assumption (A3) with  $u \equiv 1$ ,  $\delta = 1$  that gives  $C_2 = (b^2 - 1)\pi c_1(1)R_2^2$ . And assumption (A2) yields the upper bound for the areas with  $C_4 = \pi R_2^2$ .  $\square$

**Lemma 3.2.** *Let  $c > 0$  be a constant. For each  $\delta > 0$  there exists a constant  $C(\delta) > 0$  such that for all  $v \in \dot{W}_2^1(\Omega^\varepsilon, \partial\theta_0^\varepsilon)$ ,  $u \in W_2^1(\Omega)$  and all sufficiently small  $\varepsilon$  the estimates*

$$\|v\|_{L_2\left(\{x:|\tau|<c\varepsilon\eta\}\setminus\bigcup_{k\in\mathbb{M}_1}B_{R_2\varepsilon\eta}(y_k^\varepsilon)\right)} \leq \varepsilon^{1/2}\eta^{1/2}(\delta\|\nabla v\|_{L_2(\Omega^\varepsilon)} + C(\delta)\|v\|_{L_2(\Omega^\varepsilon)}), \quad (3.1)$$

$$\sum_{k\in\mathbb{M}_1} \|v\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon)\setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon))}^2 \leq \varepsilon\eta^2(|\ln\eta| + 1)(\delta\|\nabla v\|_{L_2(\Omega^\varepsilon)}^2 + C(\delta)\|v\|_{L_2(\Omega^\varepsilon)}^2), \quad (3.2)$$

$$\sum_{k\in\mathbb{Z}} \|u\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon))}^2 \leq \varepsilon\eta^2(|\ln\eta| + 1)(\delta\|\nabla u\|_{L_2(\Omega)}^2 + C(\delta)\|u\|_{L_2(\Omega)}^2) \quad (3.3)$$

*hold true. There exists a constant  $C > 0$  such that for all  $v \in W_2^1(\Omega^\varepsilon)$  and all sufficiently small  $\varepsilon$  the estimate*

$$\|v\|_{L_2(\{x:\tau=-(b+1)R_2\varepsilon\})} \leq C\|v\|_{W_2^1(\Omega^\varepsilon)} \quad (3.4)$$

*is valid.*

*Proof.* Since the function  $v$  vanishes on  $\partial\theta_0^\varepsilon$ , we extend it by zero inside  $\theta_0^\varepsilon$  and after that it belongs to  $W_2^1(\Omega \setminus \theta_1^\varepsilon)$  and has the same  $L_2$ - and  $W_2^1$ -norm. By  $\chi_1 = \chi_1(t)$  we denote an infinitely differentiable cut-off function being one as  $t < 1$  and vanishing as  $t > 2$ . Then for sufficiently small  $\delta$  and  $|\tau| < \tau_0/4$

$$v(\tau, s) = \int_{\pm \frac{\delta\tau_0}{2}}^{\tau} \frac{\partial}{\partial t} \left( v(t, s) \chi_1 \left( \frac{4|t|}{\delta\tau_0} \right) \right) dt, \quad \pm\tau > 0,$$

and by Cauchy-Schwarz inequality

$$|v(\tau, s)|^2 \leq \frac{\delta^2 \tau_0^2}{2} \int_{\pm \frac{\delta \tau_0}{2}}^{\tau} \left| \frac{\partial v}{\partial \tau}(t, s) \right|^2 dt + C(\delta) \int_{\pm \frac{\delta \tau_0}{2}}^{\tau} |v(t, s)|^2 dt, \quad \pm \tau > 0. \quad (3.5)$$

Integrating these estimates over  $\{x : 0 < \pm \tau < C\varepsilon\eta\} \setminus \bigcup_{k \in \mathbb{M}_1} B_{R_2\varepsilon\eta}(y_k^\varepsilon)$  and  $\{x : \tau = -(b+1)R_2\varepsilon\}$ , we arrive at (3.1), (3.4).

We have

$$v(x) = v(x)\chi_1 \left( \frac{|x - y_k^\varepsilon| R_2^{-1} \varepsilon^{-1} - 1}{b-1} \right) \quad \text{as } x \in B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon).$$

Let  $(r, \varphi)$  be polar coordinates centered at  $y_k^\varepsilon$ . By assumption (A2) the ball  $B_{(2b-1)R_2\varepsilon}(y_k^\varepsilon)$  does not intersect with  $\omega_i^\varepsilon$ ,  $i \neq k$ . Hence, for  $R_2\varepsilon\eta \leq r \leq bR_2\varepsilon\eta$  the Cauchy-Schwarz inequality implies

$$\begin{aligned} |v(r, \varphi)|^2 &= \left| \int_r^{(2b-1)R_2\varepsilon} \frac{\partial}{\partial t} \left( v(t, \varphi) \chi_1 \left( \frac{|x - y_k^\varepsilon| R_2^{-1} \varepsilon^{-1} - 1}{b-1} \right) \right) dt \right|^2 \\ &\leq \int_r^{(2b-1)R_2\varepsilon} \frac{dt}{t} \int_r^{(2b-1)R_2\varepsilon} \left| \frac{\partial}{\partial t} \left( v(t, \varphi) \chi_1 \left( \frac{|x - y_k^\varepsilon| R_2^{-1} \varepsilon^{-1} - 1}{b-1} \right) \right) \right|^2 t dt \quad (3.6) \\ &\leq C(|\ln \eta| + 1) \int_{R_2\varepsilon\eta}^{(2b-1)R_2\varepsilon} \left( \left| \frac{\partial u}{\partial r}(t, \varphi) \right|^2 + \varepsilon^{-2} |v(t, \varphi)|^2 \right). \end{aligned}$$

Integrating this estimate over  $B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon)$  and summing up then over  $k \in \mathbb{Z}$ , we obtain

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} \|v\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon))}^2 \\ &\leq C\eta^2(|\ln \eta| + 1) \left( \varepsilon^2 \|\nabla v\|_{L_2(\Omega^\varepsilon)}^2 + \|v\|_{L_2\left(\{x: |\tau| < (2b-1)R_2\varepsilon\} \setminus \bigcup_{k \in \mathbb{Z}} B_{R_2\varepsilon\eta}(y_k^\varepsilon)\right)}^2 \right). \end{aligned}$$

This estimate together with (3.1) lead us to (3.2). Estimate (3.3) is proven in the same way.  $\square$

*Remark 3.1.* The main idea of the proof of (3.2) is borrowed from that of Lemma 3.2 in [36].

**Lemma 3.3.** *There exists a constant  $C > 0$  such that for each  $v \in W_2^1(\Omega^\varepsilon)$  and for all sufficiently small  $\varepsilon$  the estimate*

$$\|v\|_{L_2(\partial\theta_1^\varepsilon)}^2 \leq C\eta(|\ln \eta| + 1)(\delta \|\nabla v\|_{L_2(\Omega^\varepsilon)}^2 + C(\delta) \|v\|_{L_2(\Omega^\varepsilon)}^2)$$

*holds true.*

*Proof.* Due to assumption (A2) we can cover the set  $\theta_1^\varepsilon$  by the union of the balls  $\bigcup_{k \in \mathbb{M}_1} B_{bR_2\varepsilon\eta}(y_k^\varepsilon)$ . Each point of  $\Omega^\varepsilon$  belongs to finite number of the balls  $B_{bR_2\varepsilon\eta}(y_k^\varepsilon)$  and this number is uniformly bounded in  $\varepsilon$  and all points. Rescaling each  $\omega_k^\varepsilon$  in  $(\varepsilon\eta)^{-1}$  times and employing assumption (A3) and (3.1), we get

$$\|v\|_{L_2(\partial\theta_1^\varepsilon)}^2 = \sum_{k \in \mathbb{M}_1} \|v\|_{L_2(\partial\omega_k^\varepsilon)}^2 \leq \delta\varepsilon\eta \sum_{k \in \mathbb{M}_1} \|\nabla v\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2$$

$$+ C(\delta)\varepsilon^{-1}\eta^{-1} \sum_{k \in \mathbb{M}_1} \|v\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon))}^2.$$

It remains to employ (3.2) to complete the proof.  $\square$

**Lemma 3.4.** *For any  $\delta > 0$  there exists a constant  $C(\delta) > 0$  such that the estimate*

$$|(av, v)_{L_2(\gamma)}| \leq \delta \|\nabla v\|_{L_2(\Omega)}^2 + C(\delta) \|v\|_{L_2(\Omega)}^2$$

*is valid for all  $v \in W_2^1(\Omega)$ . The constant  $C(\delta)$  is independent of  $v$ .*

The statement of this lemma follows from [30, Ch. II, Sec. 2, Ineq. (2.38)].

**Lemma 3.5.** *The estimates*

$$\begin{aligned} \|u\|_{W_2^1(\Omega)}^2 &\leq C \left( \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right)_{L_2(\Omega)} + \sum_{j=1}^2 \left( A_j \frac{\partial u}{\partial x_j}, u \right)_{L_2(\Omega)} \right. \\ &\quad \left. + \sum_{j=1}^2 \left( u, A_j \frac{\partial u}{\partial x_j} \right)_{L_2(\Omega)} + (A_0 u, u)_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}^2 \right), \\ \|u\|_{W_2^1(\Omega)}^2 &\leq C \left( \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right)_{L_2(\Omega)} + \sum_{j=1}^2 \left( A_j \frac{\partial u}{\partial x_j}, u \right)_{L_2(\Omega)} \right. \\ &\quad \left. + \sum_{j=1}^2 \left( u, A_j \frac{\partial u}{\partial x_j} \right)_{L_2(\Omega)} + (A_0 u, u)_{L_2(\Omega)} \right. \\ &\quad \left. + (\beta u, u)_{L_2(\gamma)} + \|u\|_{L_2(\Omega)}^2 \right), \\ \|v\|_{W_2^1(\Omega^\varepsilon)}^2 &\leq C \left( \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega^\varepsilon)} + \sum_{j=1}^2 \left( A_j \frac{\partial v}{\partial x_j}, v \right)_{L_2(\Omega^\varepsilon)} \right. \\ &\quad \left. + \sum_{j=1}^2 \left( v, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega^\varepsilon)} + (A_0 v, v)_{L_2(\Omega^\varepsilon)} \right. \\ &\quad \left. + (av, v)_{L_2(\partial\theta_\varepsilon^0)} + \|v\|_{L_2(\Omega^\varepsilon)}^2 \right) \end{aligned} \quad (3.7)$$

hold true for all  $u \in W_2^1(\Omega)$ ,  $v \in W_2^1(\Omega^\varepsilon)$ , where  $\beta$  is any function in  $W_\infty^1(\gamma)$ .

*Proof.* The estimates follow directly from the ellipticity condition in (2.2), Lemma 3.3, and Cauchy-Schwarz inequality.  $\square$

**Lemma 3.6.** *The estimates*

$$\begin{aligned} \|(\mathcal{H}^\varepsilon - i)^{-1} f\|_{W_2^1(\Omega^\varepsilon)} &\leq C \|f\|_{L_2(\Omega^\varepsilon)}, \\ \|(\mathcal{H}_D^0 - i)^{-1} f\|_{W_2^2(\Omega_\pm)} &\leq C \|f\|_{L_2(\Omega_\pm)}, \\ \|(\mathcal{H}^0 - i)^{-1} f\|_{W_2^2(\Omega)} &\leq C \|f\|_{L_2(\Omega)}, \end{aligned} \quad (3.8)$$

$$\|(\mathcal{H}_\beta^0 - i)^{-1} f\|_{W_2^1(\Omega \setminus \gamma)} \leq C(\|\beta\|_{W_\infty^1(\gamma)} + 1) \|f\|_{L_2(\Omega)} \quad (3.9)$$

hold true, where  $\beta \in W_\infty^1(\gamma)$ .

*Proof.* The first estimate is implied by Lemma 3.5 and the integral identity for  $(\mathcal{H}^\varepsilon - i)^{-1} f$ . And the three last estimates can be proven completely in the same way as Lemma 8.1 in [30, Ch. III, Sec. 8].  $\square$

## 4 Dirichlet condition on boundaries of holes: homogenized Dirichlet condition

In this section we prove Theorem 2.1. Let  $f \in L_2(\Omega)$ ,  $u^\varepsilon := (\mathcal{H}^\varepsilon - i)^{-1}f$ ,  $u^0 := (\mathcal{H}_D^0 - i)^{-1}f$ ,  $v^\varepsilon := u^\varepsilon - (1 - \chi_1^\varepsilon)u^0$ ,  $\chi_1^\varepsilon(x) := \chi_1\left(\frac{|\tau|}{R_2^\varepsilon}\right)$  as  $|\tau| < \tau_0$  and  $\chi_1^\varepsilon(x) := 0$  outside the set  $\{x : |\tau| < \tau_0\}$ . Here  $\chi_1$  is the cut-off function introduced in the proof of Lemma 3.2. The definition of the operators  $\mathcal{H}^\varepsilon$  and  $\mathcal{H}_D^0$  yield the integral identities

$$\begin{aligned} \mathfrak{h}^\varepsilon(u^\varepsilon, v^\varepsilon) - i(u^\varepsilon, v^\varepsilon)_{L_2(\Omega^\varepsilon)} &= (f, v^\varepsilon)_{L_2(\Omega^\varepsilon)}, \\ \mathfrak{h}_D^0(u^0, (1 - \chi_1^\varepsilon)v^\varepsilon) - i(u^0, (1 - \chi_1^\varepsilon)v^\varepsilon)_{L_2(\Omega)} &= (f, (1 - \chi_1^\varepsilon)v^\varepsilon)_{L_2(\Omega)}. \end{aligned} \quad (4.1)$$

Since  $1 - \chi_1^\varepsilon$  vanishes on each  $\omega_k^\varepsilon$ , we have

$$\begin{aligned} (u^0, (1 - \chi_1^\varepsilon)v^\varepsilon)_{L_2(\Omega)} &= (u^0, (1 - \chi_1^\varepsilon)v^\varepsilon)_{L_2(\Omega^\varepsilon)}, \\ (f, (1 - \chi_1^\varepsilon)v^\varepsilon)_{L_2(\Omega)} &= (f, (1 - \chi_1^\varepsilon)v^\varepsilon)_{L_2(\Omega^\varepsilon)}, \\ (a(1 - \chi_1^\varepsilon)u^0, v^\varepsilon)_{L_2(\partial\theta_0^\varepsilon)} &= 0, \end{aligned} \quad (4.2)$$

and by the definition of  $\mathfrak{h}_D^0$

$$\begin{aligned} \mathfrak{h}_D^0(u^0, (1 - \chi_1^\varepsilon)v^\varepsilon) &= \mathfrak{h}_D^0((1 - \chi_1^\varepsilon)u^0, v^\varepsilon) - \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u^0}{\partial x_j} \frac{\partial \chi_1^\varepsilon}{\partial x_i}, v^\varepsilon \right)_{L_2(\Omega^\varepsilon)} \\ &\quad + \sum_{i,j=1}^2 \left( A_{ij} u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_j}, \frac{\partial v^\varepsilon}{\partial x_i} \right)_{L_2(\Omega^\varepsilon)} + \sum_{j=1}^2 \left( A_j u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_j}, v^\varepsilon \right)_{L_2(\Omega^\varepsilon)} \\ &\quad - \sum_{j=1}^2 \left( u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_j}, A_j v^\varepsilon \right)_{L_2(\Omega^\varepsilon)}. \end{aligned} \quad (4.3)$$

We deduct the formulae in (4.1) one from the other and employ (4.2), (4.3),

$$\begin{aligned} \mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon) - i\|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 &= (\chi_1^\varepsilon f, v^\varepsilon)_{L_2(\Omega^\varepsilon)} - \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u^0}{\partial x_j} \frac{\partial \chi_1^\varepsilon}{\partial x_i}, v^\varepsilon \right)_{L_2(\Omega^\varepsilon)} \\ &\quad + \sum_{i,j=1}^2 \left( A_{ij} u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_j}, \frac{\partial v^\varepsilon}{\partial x_i} \right)_{L_2(\Omega^\varepsilon)} + \sum_{j=1}^2 \left( A_j u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_j}, v^\varepsilon \right)_{L_2(\Omega^\varepsilon)} \\ &\quad - \sum_{j=1}^2 \left( u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_j}, A_j v^\varepsilon \right)_{L_2(\Omega^\varepsilon)}. \end{aligned} \quad (4.4)$$

Our next step is to estimate the right hand side of the latter identity. In order to do it, we need several auxiliary lemmata.

**Lemma 4.1.** *For any  $u \in \mathfrak{D}(\mathcal{H}_D^0)$  and  $|\tau| < \tau_0/3$  the estimates*

$$\begin{aligned} |u(s, \tau)|^2 &\leq C\tau^2 \|u(s, \cdot)\|_{W_2^2(-\frac{\tau_0}{2}, \frac{\tau_0}{2})}^2, \\ |\nabla_{s,\tau} u(s, \tau)|^2 &\leq C \|\nabla_{s,\tau} u(s, \cdot)\|_{W_2^1(-\frac{\tau_0}{2}, \frac{\tau_0}{2})}^2 \end{aligned}$$

hold true, where  $C$  are the constants independent of  $u$ ,  $s$ , and  $\tau$ .

*Proof.* The desired estimates follow from the obvious relation

$$|u(s, \tau)|^2 = \left| \int_0^\tau \frac{\partial u}{\partial \tau}(s, t) dt \right|^2 \leq |\tau| \int_{-\frac{\tau_0}{2}}^{\frac{\tau_0}{2}} \left| \frac{\partial u}{\partial \tau}(s, t) \right|^2 dt \quad (4.5)$$

and (3.5). □

Denote  $\Pi^\varepsilon := \{x : |\tau| < c_3\varepsilon\}$  and assume that

$$c_3 \geq bR_2, \quad \overline{\bigcup_{k \in \mathbb{M}_0} B_{R_3\varepsilon}(y_k^\varepsilon)} \subset \Pi^\varepsilon, \quad \overline{\bigcup_{k \in \mathbb{M}} B_{bR_2\varepsilon\eta}(y_k^\varepsilon)} \subset \Pi^\varepsilon. \quad (4.6)$$

**Lemma 4.2.** *The estimate*

$$\|v^\varepsilon\|_{L_2(\Pi^\varepsilon \setminus \theta^\varepsilon)} \leq C\varepsilon(|\ln \eta(\varepsilon)|^{1/2} + 1)\|\nabla v^\varepsilon\|_{L_2(\Omega^\varepsilon)}$$

holds true.

*Proof.* We extend the function  $v^\varepsilon$  by zero inside  $\theta_0^\varepsilon$ . Since  $v^\varepsilon$  vanishes on  $\partial\theta_\varepsilon^0$ , the extension belongs to  $W_2^1(\Omega \setminus \theta_1^\varepsilon)$  and has the same  $L_2$ - and  $W_2^1$ -norm.

By assumption (A2) for each  $k \in \mathbb{M}_0$  the ball  $B_{R_1\varepsilon\eta}(x^k + y_k^\varepsilon)$  lies inside  $\omega_k^\varepsilon$ . We introduce polar coordinates  $(r, \theta)$  centered at  $x^k + y_k^\varepsilon$ . Since  $v^\varepsilon = 0$  as  $r = R_1\varepsilon\eta$ , we have

$$v^\varepsilon(x) = \int_{R_1\varepsilon\eta}^r \frac{\partial v^\varepsilon}{\partial r} dr, \quad |v^\varepsilon(x)|^2 \leq \ln \frac{r}{R_1\varepsilon\eta} \int_{R_1\varepsilon\eta}^r \left| \frac{\partial v^\varepsilon}{\partial r} \right|^2 r dr. \quad (4.7)$$

Integrating this estimate, we get

$$\begin{aligned} & \|v^\varepsilon\|_{L_2(\{x: s_{k,-}^\varepsilon < s < s_{k,+}^\varepsilon, |\tau| < (c_3+1)\varepsilon\})}^2 + \|v^\varepsilon\|_{L_2(\{x: s = s_{k,\pm}^\varepsilon, |\tau| < (c_3+1)\varepsilon\})}^2 \\ & \leq C\varepsilon^2(|\ln \eta| + 1)\|\nabla v^\varepsilon\|_{L_2(\{x: s_{k,-}^\varepsilon < s < s_{k,+}^\varepsilon, |\tau| < (c_3+1)\varepsilon\})}^2, \end{aligned} \quad (4.8)$$

where  $s_{k,\pm}^\varepsilon$  are chosen so that the sets  $\{x : s = s_{k,\pm}^\varepsilon, |\tau| < (c_3 + 1)\varepsilon\}$  do not intersect with  $\theta^\varepsilon$ , and for each  $k \in \mathbb{Z}$

$$B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \subset \{x : s_{k,-}^\varepsilon < s < s_{k,+}^\varepsilon, |\tau| < (c_3 + 1)\varepsilon\}.$$

Let  $\chi_2^\varepsilon = \chi_2^\varepsilon(x)$  be an infinitely differentiable cut-off function with values in  $[0, 1]$ , being one in  $B_{R_2\varepsilon\eta}(y_k^\varepsilon)$ ,  $k \in \mathbb{M}_1$ , vanishing outside  $\bigcup_{k \in \mathbb{M}_1} B_{bR_2\varepsilon\eta}(y_k^\varepsilon)$ , and satisfying the uniform estimate  $|\nabla \chi_2^\varepsilon| \leq C\varepsilon^{-1}\eta^{-1}(\varepsilon)$ . We represent  $v^\varepsilon$  as  $v^\varepsilon = \chi_2^\varepsilon v^\varepsilon + (1 - \chi_2^\varepsilon)v^\varepsilon$  and see that  $\chi_2^\varepsilon v^\varepsilon \in \dot{W}_2^1(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon, \partial B_{bR_2\varepsilon\eta}(y_k^\varepsilon))$ . Then by assumption (A6) we have the inequalities

$$\begin{aligned} & \|v^\varepsilon\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2 \leq C\|v^\varepsilon\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon))}^2 \\ & \quad + C\varepsilon^2\eta^2\|\nabla \chi_2^\varepsilon v^\varepsilon\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2, \quad k \in \mathbb{M}_1, \\ & \varepsilon^2\eta^2\|\nabla \chi_2^\varepsilon v^\varepsilon\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon))}^2 \leq C\|v^\varepsilon\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2 \\ & \quad + C\varepsilon^2\eta^2\|\nabla v^\varepsilon\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2, \quad k \in \mathbb{M}_1. \end{aligned}$$

Hence, summing up w.r.t.  $k$ ,

$$\begin{aligned} & \|v^\varepsilon\|_{L_2\left(\bigcup_{k \in \mathbb{M}_1} B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon\right)}^2 \leq C\varepsilon^2\eta^2\|\nabla v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 \\ & \quad + C\|v^\varepsilon\|_{L_2\left(\bigcup_{k \in \mathbb{M}_1} B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon)\right)}^2. \end{aligned} \quad (4.9)$$

In view of (4.6), the domain  $\{x : c_3\varepsilon < |\tau| < (c_3 + 1)\varepsilon\}$  do not intersect with  $\bigcup_{k \in \mathbb{M}} B_{bR_2\varepsilon\eta}(y_k^\varepsilon)$ . Thus, for  $c_3\varepsilon < |\tau| < (c_3 + 1)\varepsilon$  and  $s \in (s_{k,+}^\varepsilon, s_{k+1,-}^\varepsilon)$  one has

$$v^\varepsilon(s, \tau) = v^\varepsilon(s_{k,+}^\varepsilon, \tau) + \int_{s_{k,+}^\varepsilon}^s \frac{\partial v^\varepsilon}{\partial s} ds.$$

Assumption (A5) yields the inequality  $|s_{k+1,-}^\varepsilon - s_{k,+}^\varepsilon| \leq C\varepsilon$ ,  $k \in \mathbb{M}_0$ . Thus,

$$|v^\varepsilon(s, \tau)|^2 \leq C \left( |v^\varepsilon(s_{k,+}^\varepsilon, \tau)|^2 + \varepsilon \int_{s_{k,+}^\varepsilon}^{s_{k+1,-}^\varepsilon} \left| \frac{\partial v^\varepsilon}{\partial s} \right|^2 ds \right).$$

Integrating this estimate over  $|\tau| \in (c_3\varepsilon, (c_3+1)\varepsilon)$ ,  $s \in (s_{k,+}^\varepsilon, s_{k+1,-}^\varepsilon)$  and employing (4.8), we obtain

$$\begin{aligned} \|v^\varepsilon\|_{L_2(\{x: c_3\varepsilon < |\tau| < (c_3+1)\varepsilon, s_{k,+}^\varepsilon < s < s_{k+1,-}^\varepsilon\})}^2 \\ \leq C\varepsilon^2(|\ln \eta| + 1) \|v^\varepsilon\|_{L_2(\{x: c_3\varepsilon < |\tau| < (c_3+1)\varepsilon, s_{k,+}^\varepsilon < s < s_{k+1,-}^\varepsilon\})}^2, \quad k \in \mathbb{M}_0. \end{aligned}$$

Summing up w.r.t.  $k$ , we arrive at one more estimate

$$\|v^\varepsilon\|_{L_2(\{x: c_3\varepsilon < |\tau| < (c_3+1)\varepsilon\})}^2 \leq C\varepsilon^2(|\ln \eta| + 1) \|\nabla v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2. \quad (4.10)$$

Given any  $x \in \Pi^\varepsilon \setminus \bigcup_{k \in \mathbb{M}} B_{R_2\varepsilon\eta}(y_k^\varepsilon)$ , consider the associated values  $(s, \tau)$ . We then have

$$v^\varepsilon(x) = \chi_1^\varepsilon(x) v^\varepsilon(x) = \int_{\pm(c_3+1)\varepsilon}^{\tau} \frac{\partial \chi_1^\varepsilon v^\varepsilon}{\partial \tau} d\tau, \quad \pm\tau > 0.$$

Hence,

$$|v^\varepsilon(x)|^2 \leq C\varepsilon^{-1} \left| \int_{\pm(c_3+1)\varepsilon}^{\tau} |v^\varepsilon|^2 d\tau \right| + C\varepsilon \left| \int_{\pm(c_3+1)\varepsilon}^{\tau} \left| \frac{\partial v^\varepsilon}{\partial \tau} \right|^2 d\tau \right|.$$

We integrate this estimate over  $\Pi^\varepsilon \setminus \bigcup_{k \in \mathbb{M}} B_{R_2\varepsilon\eta}(y_k^\varepsilon)$  and employ (4.10). It yields

$$\begin{aligned} \|v^\varepsilon\|_{L_2\left(\Pi^\varepsilon \setminus \bigcup_{k \in \mathbb{M}} B_{R_2\varepsilon\eta}(y_k^\varepsilon)\right)}^2 &\leq C \|v^\varepsilon\|_{L_2(\{x: c_3\varepsilon < \tau < (c_3+1)\varepsilon\})}^2 + C\varepsilon^2 \|\nabla v^\varepsilon\|_{L_2(\Pi^\varepsilon)}^2 \\ &\leq C\varepsilon^2(|\ln \eta| + 1) \|\nabla v^\varepsilon\|_{L_2(\Pi^\varepsilon)}^2. \end{aligned}$$

The obtained estimate, (4.9), and (4.8) imply the statement of the lemma.  $\square$

Let us estimate the right hand side of (4.4). By Lemma 4.2 we get

$$\begin{aligned} |(1 - \chi_1^\varepsilon)f, v^\varepsilon|_{L_2(\Omega^\varepsilon)} &= |(f, (1 - \chi_1^\varepsilon)v^\varepsilon)|_{L_2(\Pi^\varepsilon \setminus \theta^\varepsilon)} \leq C \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{L_2(\Pi^\varepsilon \setminus \theta^\varepsilon)} \\ &\leq C\varepsilon(|\ln \eta(\varepsilon)|^{1/2} + 1) \|f\|_{L_2(\Omega)} \|\nabla v^\varepsilon\|_{L_2(\Omega^\varepsilon)}. \end{aligned}$$

In the same fashion employing Lemmata 3.5, 3.6, 4.1, 4.2, we obtain the estimate for the next term in the right hand side of (4.4),

$$\begin{aligned} \left| \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u^0}{\partial x_j}, v^\varepsilon \frac{\partial \chi_1^\varepsilon}{\partial x_i} \right) \right|_{L_2(\Omega^\varepsilon)} &\leq C\varepsilon^{-1} \|\nabla u^0\|_{L_2(\Omega^\varepsilon)} \|v^\varepsilon\|_{L_2(\Omega^\varepsilon \setminus \theta^\varepsilon)} \\ &\leq C\varepsilon^{1/2}(|\ln \eta(\varepsilon)|^{1/2} + 1) \|u^0\|_{W_2^2(\Omega)} \|\nabla v^\varepsilon\|_{L_2(\Omega^\varepsilon)} \\ &\leq C\varepsilon^{1/2}(|\ln \eta(\varepsilon)|^{1/2} + 1) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned}$$

The other terms in the right hand side of (4.4) are estimated in the same way,

$$\begin{aligned} \left| \sum_{i,j=1}^2 \left( A_{ij} u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_j}, \frac{\partial v^\varepsilon}{\partial x_i} \right) \right|_{L_2(\Omega^\varepsilon)} &\leq C\varepsilon^{-1} \|u^0\|_{L_2(\Omega^\varepsilon)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \\ &\leq C\varepsilon^{1/2} \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}, \end{aligned}$$



$$\begin{aligned}
& \left| \sum_{j=1}^2 \left( A_j u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_j}, v^\varepsilon \right)_{L_2(\Omega^\varepsilon)} - \sum_{j=1}^2 \left( u^0, A_j v^\varepsilon \frac{\partial \chi_1^\varepsilon}{\partial x_j} \right)_{L_2(\Omega^\varepsilon)} \right| \\
& \leq C \varepsilon^{-1} \|u^0\|_{L_2(\Omega)} \|v^\varepsilon\|_{L_2(\Pi^\varepsilon \setminus \theta^\varepsilon)} \\
& \leq C \varepsilon^{3/2} (|\ln \eta(\varepsilon)|^{1/2} + 1) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}.
\end{aligned}$$

Substituting last four estimates into the right hand side of (4.4) and using Lemma 3.5, we arrive at the inequality

$$\|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \leq C \varepsilon^{1/2} (|\ln \eta(\varepsilon)|^{1/2} + 1) \|f\|_{L_2(\Omega)}.$$

It remains to estimate the norm  $\|\chi_1^\varepsilon u^0\|_{W_2^1(\Omega^\varepsilon)}$  to complete the proof. Employing Lemmata 3.6, 4.1, one can check easily that

$$\begin{aligned}
\|\chi_1^\varepsilon u^0\|_{L_2(\Omega^\varepsilon)} & \leq C \varepsilon^{3/2} \|f\|_{L_2(\Omega)}, \\
\|\nabla \chi_1^\varepsilon u^0\|_{L_2(\Omega^\varepsilon)} & \leq C (\|\chi_1^\varepsilon \nabla u^0\|_{L_2(\Omega^\varepsilon)} + \varepsilon^{-1} \|u^0\|_{L_2(\Omega^\varepsilon)}) \leq C \varepsilon^{1/2} \|f\|_{L_2(\Omega)}.
\end{aligned}$$

The proof is complete.

## 5 Robin condition on boundaries of holes

In this section we prove Theorems 2.3, 2.4. We begin with Theorem 2.3.

### 5.1 Proof of Theorem 2.3

Let  $f \in L_2(\Omega)$ ,  $u^\varepsilon := (\mathcal{H}^\varepsilon - i)^{-1} f$ ,  $u^0 := (\mathcal{H}^0 - i)^{-1} f$ ,  $v^\varepsilon := u^\varepsilon - u^0$ . Assumption (A8) implies that  $\theta_0^\varepsilon = \emptyset$ ,  $\theta_1^\varepsilon = \theta^\varepsilon$ . Since  $u^0 \in W_2^2(\Omega)$ , by the standard embedding theorems the function  $(\frac{\partial}{\partial N^\varepsilon} + a) u^0$  belongs to  $L_2(\partial\theta^\varepsilon)$ . Then the function  $v^\varepsilon$  is the generalized solution to the boundary value problem

$$\begin{aligned}
& \left( - \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^2 A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 - i \right) v^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \\
& v^\varepsilon = 0 \quad \text{on } \partial\Omega, \quad \left( \frac{\partial}{\partial N^\varepsilon} + a \right) v^\varepsilon = - \left( \frac{\partial}{\partial N^\varepsilon} + a \right) u^0 \quad \text{on } \partial\theta^\varepsilon.
\end{aligned}$$

Taking  $v^\varepsilon$  as the test function, we write the associated integral identity

$$\mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon) - i \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 = - \left( \left( \frac{\partial}{\partial N^\varepsilon} + a \right) u^0, v^\varepsilon \right)_{L_2(\theta^\varepsilon)}. \quad (5.1)$$

The main idea of our proof is to estimate the right hand side of this identity in an appropriate way and to get then the desired estimate for  $v^\varepsilon$ .

It is clear that

$$\left| \left( \left( \frac{\partial}{\partial N^\varepsilon} + a \right) u^0, v^\varepsilon \right)_{L_2(\theta^\varepsilon)} \right| \leq \left| \left( \frac{\partial u^0}{\partial N^\varepsilon}, v^\varepsilon \right)_{L_2(\theta^\varepsilon)} \right| + C \|u^0\|_{L_2(\theta^\varepsilon)} \|v^\varepsilon\|_{L_2(\theta^\varepsilon)}. \quad (5.2)$$

If  $a \equiv 0$ , the constant  $C$  in this estimate can be taken zero. Lemma 3.3 and (3.8) imply the estimate for the last term in the right hand side of (5.2),

$$\begin{aligned}
\|u^0\|_{L_2(\theta^\varepsilon)} \|v^\varepsilon\|_{L_2(\theta^\varepsilon)} & \leq C \eta (|\ln \eta| + 1) \|u^0\|_{W_2^1(\Omega^\varepsilon)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \\
& \leq C \eta (|\ln \eta| + 1) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}.
\end{aligned} \quad (5.3)$$

We proceed to estimating the term  $\left( \frac{\partial u^0}{\partial N^\varepsilon}, v^\varepsilon \right)_{L_2(\theta^\varepsilon)}$ . We let

$$\left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial\omega_k^\varepsilon} := \frac{1}{|\partial\omega_k| \varepsilon \eta} \int_{\partial\omega_k^\varepsilon} \frac{\partial u^0}{\partial N^\varepsilon} ds,$$

$$\langle v^\varepsilon \rangle_{B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon)} := \frac{1}{(b^2 - 1)\pi R_2^2 \varepsilon^2 \eta^2} \int_{B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon)} v^\varepsilon dx,$$

$$U_k^\varepsilon := \frac{\partial u^0}{\partial N^\varepsilon} - \left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial \omega_k^\varepsilon}, \quad V_k^\varepsilon := v^\varepsilon - \langle v^\varepsilon \rangle_{B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon)}.$$

Obviously,

$$U_k^\varepsilon \in L_2(\partial \omega_k^\varepsilon), \quad \int_{\partial \omega_k^\varepsilon} U_k^\varepsilon ds = 0,$$

$$\int_{B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon)} V_k^\varepsilon dx = 0, \quad (5.4)$$

and also

$$\left( \frac{\partial u^0}{\partial N^\varepsilon}, v^\varepsilon \right)_{L_2(\theta^\varepsilon)} = \left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial \omega_k^\varepsilon} \int_{\partial \omega_k^\varepsilon} \overline{v^\varepsilon} ds + (U_k^\varepsilon, V_k^\varepsilon)_{L_2(\partial \omega_k^\varepsilon)}.$$

By Cauchy-Schwarz inequality and Lemma 3.1 it yields

$$\begin{aligned} \left| \left( \frac{\partial u^0}{\partial N^\varepsilon}, v^\varepsilon \right)_{L_2(\theta^\varepsilon)} \right| &\leq \varepsilon^{1/2} \eta^{1/2} \sum_{k \in \mathbb{Z}} |\partial \omega_k|^{1/2} \left| \left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial \omega_k^\varepsilon} \right| \|v^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)} \\ &\quad + \sum_{k \in \mathbb{Z}} \|U_k^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)} \|V_k^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)} \\ &\leq C \varepsilon^{1/2} \eta^{1/2} \left( \sum_{k \in \mathbb{Z}} \left| \left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial \omega_k^\varepsilon} \right|^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \|v^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)}^2 \right)^{1/2} \\ &\quad + \left( \sum_{k \in \mathbb{Z}} \|U_k^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)}^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \|V_k^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)}^2 \right)^{1/2}. \end{aligned} \quad (5.5)$$

Integrating by parts, we get

$$\left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial \omega_k^\varepsilon} = -\frac{1}{|\partial \omega_k| \varepsilon \eta} \int_{\omega_k^\varepsilon} \left( \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^2 \frac{\partial}{\partial x_j} \overline{A_j} \right) u^0 dx.$$

Employing Lemma 3.1 and Cauchy-Schwarz inequality, we obtain the estimate for  $\left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial \omega_k^\varepsilon}$ ,

$$\left| \left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial \omega_k^\varepsilon} \right| \leq C \|u^0\|_{W_2^2(\omega_k^\varepsilon)}. \quad (5.6)$$

Since

$$\|U_k^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)}^2 + \varepsilon \eta |\partial \omega_k| \left| \left\langle \frac{\partial u^0}{\partial N^\varepsilon} \right\rangle_{\partial \omega_k^\varepsilon} \right|^2 = \left\| \frac{\partial u^0}{\partial N^\varepsilon} \right\|_{L_2(\partial \omega_k^\varepsilon)}^2 \leq C \left( \|\nabla u^0\|_{L_2(\partial \omega_k^\varepsilon)}^2 + \|u^0\|_{L_2(\partial \omega_k^\varepsilon)}^2 \right),$$

by Lemma 3.3 we have

$$\sum_{k \in \mathbb{Z}} \|U_k^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)}^2 \leq C \eta (|\ln \eta| + 1) \|u^0\|_{W_2^2(\Omega)}^2. \quad (5.7)$$

Due to assumption (A2) with  $\delta = 1$ , (5.4), Poincaré inequality [36, Ch. I, Sec. 1, Ineq. (1.5)], and by rescaling  $\omega_k^\varepsilon$  in  $(\varepsilon \eta)^{-1}$  times we get the upper bound for  $\|V^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)}$ ,

$$\begin{aligned} \|V_k^\varepsilon\|_{L_2(\partial \omega_k^\varepsilon)}^2 &\leq C \left( \varepsilon \eta \|\nabla V_k^\varepsilon\|_{L_2(B_{R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2 + \varepsilon^{-1} \eta^{-1} \|V_k^\varepsilon\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_k^\varepsilon))}^2 \right) \\ &\leq C \varepsilon \eta \|\nabla V_k^\varepsilon\|_{L_2(B_{R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2 \leq C \varepsilon \eta \|\nabla v^\varepsilon\|_{L_2(B_{bR_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2. \end{aligned} \quad (5.8)$$

The norm  $\|v^\varepsilon\|_{L_2(\partial\omega_k^\varepsilon)}$  can be estimated by Lemma 3.3. This estimate and (5.5), (5.6), (5.7), (5.8), (3.8) imply

$$\begin{aligned} \left| \left( \frac{\partial u^0}{\partial N^\varepsilon}, v^\varepsilon \right)_{L_2(\theta^\varepsilon)} \right| &\leq C\varepsilon^{1/2} \eta(|\ln \eta|^{1/2} + 1) \|u^0\|_{W_2^2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \\ &\leq C\varepsilon^{1/2} \eta(|\ln \eta|^{1/2} + 1) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned} \quad (5.9)$$

We substitute this inequality and (5.3), (5.2) into (5.1) to obtain

$$\begin{aligned} |\mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon)| + \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 &\leq C\eta(|\ln \eta| + 1) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}, & \text{if } a \neq 0, \\ |\mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon)| + \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 &\leq C\varepsilon^{1/2} \eta(|\ln \eta|^{1/2} + 1) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}, & \text{if } a \equiv 0. \end{aligned}$$

To complete the proof, it remains to estimate the left hand side of the latter inequalities by Lemma 3.5.

*Remark 5.1.* Throughout the proof we assumed (A7). But as one can easily see, all the estimates are valid also in the case  $\eta = \text{const}$ .

## 5.2 Proof of Theorem 2.4

We let  $\tilde{\gamma} := \{x : \tau = -(b+1)R_2\varepsilon, s \in \mathbb{R}\}$  and by (A2) we see that

$$B_{bR_2\varepsilon\eta(y_k^\varepsilon)} \cap \tilde{\gamma} = \emptyset$$

for all  $k \in \mathbb{Z}$  and sufficiently small  $\varepsilon$ . Given  $\beta \in W_\infty^1(\tilde{\gamma})$ , by  $\tilde{\mathcal{H}}_{\alpha a}^0$  we denote the operator with the differential expression (2.3) subject to the boundary conditions

$$[u]_{\tilde{\gamma}} = 0, \quad \left[ \frac{\partial u}{\partial \tilde{N}^0} \right]_{\tilde{\gamma}} + \beta u|_{\tilde{\gamma}} = 0, \quad (5.10)$$

$$\frac{\partial}{\partial \tilde{N}^0} := \sum_{i,j=1}^2 A_{ij} \nu_i^0 \frac{\partial}{\partial x_j}, \quad [u]_{\tilde{\gamma}} := u|_{\tau=-(b+1)R_2\varepsilon+0} - u|_{\tau=-(b+1)R_2\varepsilon-0}.$$

Here the function  $\alpha$  is defined on  $\tilde{\gamma}$  in the sense that  $\alpha = \alpha(s)$  at the point  $x = \rho(s) - (b+1)R_2\varepsilon\nu^0(s) \in \tilde{\gamma}$ . We observe that the normal to  $\tilde{\gamma}$  coincides with  $\nu^0$  and this is why exactly this vector appears in boundary conditions (5.10). The associated form is

$$\begin{aligned} \tilde{\mathfrak{h}}_\alpha^0(u, v) &:= \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega)} + \sum_{j=1}^2 \left( A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega)} \\ &\quad + \sum_{j=1}^2 \left( u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega)} + (A_0 u, v)_{L_2(\Omega)} + (\alpha a u, v)_{L_2(\tilde{\gamma})} \end{aligned}$$

in  $L_2(\Omega)$  on  $\mathring{W}_2^1(\Omega)$ .

As for  $\gamma$ , the curve  $\tilde{\gamma}$  divides  $\Omega$  into two disjoint subdomains  $\tilde{\Omega}_\pm$ , where  $\tilde{\Omega}_+$  is the upper one. By analogy with [5, Lem. 2.2], [35, Ch. IV, Sec. 2.2, 2.3], [12, Lem. 3.2] one can check that

$$\mathfrak{D}(\tilde{\mathcal{H}}_\alpha^0) = \{u \in \mathring{W}_2^1(\Omega) : u \in W_2^2(\tilde{\Omega}_\pm) \text{ and (5.10) is satisfied}\}.$$

Given  $f \in L_2(\Omega)$ , we let  $u^\varepsilon := (\mathcal{H}^\varepsilon - i)^{-1}f$ ,  $u^0 := (\mathcal{H}_\alpha^0 - i)^{-1}f$ ,  $\tilde{u}_0 := (\tilde{\mathcal{H}}_\alpha^0 - i)^{-1}f$ ,  $\hat{u}_0 := u^0 - \tilde{u}_0$ .

We first prove several auxiliary lemmata. The first one is implied by assumption (A10).

**Lemma 5.1.** *The function  $\alpha_\varepsilon(s)$  is bounded uniformly in sufficiently small  $\varepsilon$  and  $s \in \mathbb{R}$ .*

**Lemma 5.2.** Let  $\beta \in W_\infty^1(\{x : |\tau| < \tau_0/2\})$ . Then for any  $f \in L_2(\Omega)$  and all sufficiently small  $\varepsilon$  the estimates

$$\|(\tilde{\mathcal{H}}_\beta^0 - i)^{-1}f\|_{W_2^2(\Omega \setminus \tilde{\gamma})} \leq C\|f\|_{L_2(\Omega)}, \quad (5.11)$$

$$\|(\tilde{\mathcal{H}}_\beta^0 - i)^{-1}f - (\mathcal{H}_\beta^0 - i)^{-1}f\|_{W_2^2(\Omega \setminus (\gamma \cup \tilde{\gamma}))} \leq C\varepsilon^{1/2}\|f\|_{L_2(\Omega)} \quad (5.12)$$

hold true, where  $C$  are positive constants independent of  $f$ ,  $\beta$ , and  $\varepsilon$ .

*Remark 5.2.* The function  $\beta$  in boundary condition (5.10) for the operator  $\tilde{\mathcal{H}}_\beta^0$  is understood in the sense of the trace of  $\beta$  on  $\tilde{\gamma}$ .

*Proof.* The first estimate can be proven by reproducing the arguments of the proof of Lemma 8.1 in [30, Ch. III, Sec. 8] and keeping track of the dependence on  $\beta$ . Although now the operator depends on  $\varepsilon$ , the only dependence is in the definition of the curve  $\tilde{\gamma}$  and its equation depends on  $\varepsilon$  smoothly. Exactly this fact implies that the estimate (5.11) is uniform in  $\varepsilon$ .

We write the integral identities for  $u^0$  and  $\tilde{u}^0$  choosing  $\hat{u}_0 := u^0 - \tilde{u}^0$  as the test function and deduct one identity from the other. It yields

$$\mathfrak{h}_\beta^0(\hat{u}_0, \hat{u}_0) - i\|\hat{u}_0\|_{L_2(\Omega)}^2 = (\beta\tilde{u}_0, \hat{u}_0)_{\tilde{\gamma}} - (\beta\tilde{u}_0, \hat{u}_0)_\gamma. \quad (5.13)$$

It is easy to see that  $\frac{d}{ds}\nu^0(s) = K(s)\rho'(s)$ , where  $K$  is an uniformly bounded on  $\gamma$  function. Then we can rewrite the right hand side of (5.13) as

$$\begin{aligned} & (\beta\tilde{u}_0, \hat{u}_0)_{\tilde{\gamma}} - (\beta\tilde{u}_0, \hat{u}_0)_\gamma \\ &= \int_{\mathbb{R}} (\beta\tilde{u}_0\hat{u}_0)|_{\tau=-(b+1)R_2\varepsilon} (1 - (b+1)R_2\varepsilon K(s)) ds - \int_{\mathbb{R}} (\beta\tilde{u}_0\hat{u}_0)|_{\tau=0} ds \\ &= -(b+1)R_2\varepsilon \int_{\mathbb{R}} (\beta\tilde{u}_0\hat{u}_0)|_{\tau=-(b+1)R_2\varepsilon} ds - \int_{\mathbb{R}} \int_{-(b+1)R_2\varepsilon}^0 \frac{\partial}{\partial \tau} \beta\tilde{u}_0\hat{u}_0 d\tau ds. \end{aligned}$$

Lemma 3.2 and the smoothness of  $a$  and  $\alpha$  imply

$$\left| \int_{\mathbb{R}} \int_{-(b+1)R_2\varepsilon}^0 \frac{\partial}{\partial \tau} \beta\tilde{u}_0\hat{u}_0 d\tau ds \right| \leq C\varepsilon^{1/2}\|\tilde{u}_0\|_{W_2^1(\Omega)}\|\hat{u}_0\|_{W_2^1(\Omega)}.$$

Two last relations, (5.13), and Lemma 3.5 yield

$$\|\hat{u}^0\|_{W_2^1(\Omega)} \leq C\varepsilon^{1/2}\|f\|_{L_2(\Omega)}.$$

It remains to estimate  $L_2(\Omega)$ -norm of second derivatives of  $\hat{u}^0$ . We again reproduce the arguments in the proof of Lemma 8.1 in [30, Ch. III, Sec. 8]. It leads us to the estimate

$$\begin{aligned} \|\hat{u}^0\|_{W_2^2(\Omega \setminus (\gamma \cup \tilde{\gamma}))} &\leq C(\|b\|_{W_\infty^1(\{x: |\tau| < \tau_0/2\})} + 1)\|\hat{u}^0\|_{W_2^1(\Omega)} \\ &+ \left| \int_{\gamma} \frac{\partial \hat{u}^0}{\partial s} \left( P_1 \frac{\partial u^0}{\partial s} + P_2 u^0 \right) ds - \int_{\tilde{\gamma}} \frac{\partial \hat{u}^0}{\partial s} \left( P_1 \frac{\partial u^0}{\partial s} + P_2 u^0 \right) ds \right| \end{aligned} \quad (5.14)$$

where  $P_i \in W_\infty^1(x : |\tau| < \tau_0/2)$  are certain functions obeying the inequality

$$\|P_1\|_{W_\infty^1(x: |\tau| < \tau_0/2)} + \|P_2\|_{W_\infty^1(x: |\tau| < \tau_0/2)} \leq C(\|b\|_{W_\infty^1(\{x: |\tau| < \tau_0/2\})} + 1),$$

and  $C$  are constants independent of  $f$ ,  $\beta$ , and  $\varepsilon$ . We rewrite the last term in the right hand side of (5.14) as

$$\int_{\gamma} \frac{\partial \hat{u}^0}{\partial s} \left( P_1 \frac{\partial u^0}{\partial s} + P_2 u^0 \right) ds - \int_{\tilde{\gamma}} \frac{\partial \hat{u}^0}{\partial s} \left( P_1 \frac{\partial u^0}{\partial s} + P_2 u^0 \right) ds$$

$$= \int_{\gamma} \int_{-(b+1)R_2\varepsilon}^0 \frac{\partial}{\partial \tau} \left( \frac{\partial \hat{u}^0}{\partial s} \left( P_1 \frac{\partial u^0}{\partial s} + P_2 u^0 \right) \right) d\tau,$$

and employ Lemmata 5.2, 3.6 to estimate it,

$$\begin{aligned} \left| \int_{\gamma} \int_{-(b+1)R_2\varepsilon}^0 \frac{\partial}{\partial \tau} \left( \frac{\partial \hat{u}^0}{\partial s} \left( P_1 \frac{\partial u^0}{\partial s} + P_2 u^0 \right) \right) d\tau \right| &\leq C(\|b\|_{W_\infty^1(\{x:|\tau|<\tau_0/2\})} + 1) \\ &\quad \left( \left\| \frac{\partial^2 \hat{u}^0}{\partial s \partial \tau} \right\|_{L_2(\{x:-(b+1)R_2\varepsilon < \tau < 0\})} \|u^0\|_{W_2^1(\{x:-(b+1)R_2\varepsilon < \tau < 0\})} \right. \\ &\quad \left. + \left\| \frac{\partial \hat{u}^0}{\partial s} \right\|_{L_2(\{x:-(b+1)R_2\varepsilon < \tau < 0\})} \|u^0\|_{W_2^2(\{x:-(b+1)R_2\varepsilon < \tau < 0\})} \right) \\ &\leq C\varepsilon^{1/2} \|\hat{u}^0\|_{W_2^2(\{x:-(b+1)R_2\varepsilon < \tau < 0\})} \|u^0\|_{W_2^2(\{x:-(b+1)R_2\varepsilon < \tau < 0\})}. \end{aligned}$$

Combining two last relations and (5.14), we complete the proof.  $\square$

In view of Lemma 5.2, to prove the theorem it is sufficient to estimate the  $W_2^1(\Omega^\varepsilon)$ -norm of the function  $v^\varepsilon := u^\varepsilon - \hat{u}^0$ . This function solves the boundary value problem

$$\begin{aligned} \left( - \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^2 A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 - i \right) v^\varepsilon &= 0 \quad \text{in } \Omega^\varepsilon, \\ v^\varepsilon &= 0 \quad \text{on } \partial\Omega, \quad \left( \frac{\partial}{\partial N^\varepsilon} + a \right) v^\varepsilon = - \left( \frac{\partial}{\partial N^\varepsilon} + a \right) u^0 \quad \text{on } \partial\theta^\varepsilon, \\ [v^\varepsilon]_{\tilde{\gamma}} &= 0, \quad \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^0} \right]_{\tilde{\gamma}} + \alpha a \tilde{u}^0|_{\tilde{\gamma}} = 0. \end{aligned}$$

We write the associated integral identity with  $v^\varepsilon$  as the test function,

$$\mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon) - i \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 = - \left( \frac{\partial \tilde{u}^0}{\partial N^\varepsilon}, v^\varepsilon \right)_{L_2(\partial\theta^\varepsilon)} - (a \tilde{u}^0, v^\varepsilon)_{L_2(\partial\theta^\varepsilon)} + (\alpha a \tilde{u}^0, v^\varepsilon)_{L_2(\tilde{\gamma})}. \quad (5.15)$$

Let us estimate the right hand side of this identity.

Proceeding as in the proof of (5.9) in the previous section and employing (5.11) instead of (3.9), we obtain

$$\begin{aligned} \left| \left( \frac{\partial \tilde{u}^0}{\partial N^\varepsilon}, v^\varepsilon \right)_{L_2(\partial\theta^\varepsilon)} \right| &\leq C\varepsilon^{1/2} \eta^{1/2} (|\ln \eta|^{1/2} + 1) \|u^0\|_{W_2^2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \\ &\leq C\varepsilon^{1/2} \eta (|\ln \eta|^{1/2} + 1) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned} \quad (5.16)$$

To estimate the remaining terms in the right hand side of (5.15), we first use assumption (A11). For each  $v \in W_2^1(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)$  the function  $X_k^\varepsilon(x) := X_k\left(\frac{x-y_k^\varepsilon}{\varepsilon\eta}\right)$  obeys the integral identity

$$\varepsilon(\nabla X_k^\varepsilon, \nabla v)_{L_2(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)} + \int_{\partial\omega_k^\varepsilon} \bar{v} d\omega - \frac{|\partial\omega_k|}{\pi(b+1)R_2} \int_{\partial B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon)} \bar{v} d\omega = 0.$$

We choose the test function as  $v = a\tilde{u}^0 v_*$ ,  $v_* \in C^1(\overline{B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon})$  to get

$$\varepsilon(\nabla X_k^\varepsilon, a\tilde{u}^0 \nabla v_* + v_* \nabla a\tilde{u}^0)_{L_2(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)} + (a\tilde{u}^0, v_*)_{L_2(\partial\omega_k^\varepsilon)}$$

$$-\frac{|\partial\omega_k|}{\pi(b+1)R_2}(a\tilde{u}^0, v_*)_{L_2(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon))} = 0.$$

Since the space  $C^1(\overline{B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon)} \setminus \omega_k^\varepsilon)$  is dense in  $W_2^1(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)$ , the last identity is valid also for each  $v_* \in W_2^1(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)$ , and, in particular, for  $v_* = v^\varepsilon$ . It yields

$$\begin{aligned} (a\tilde{u}^0, v^\varepsilon)_{L_2(\partial\theta^\varepsilon)} &= \sum_{k \in \mathbb{Z}} (a\tilde{u}^0, v^\varepsilon)_{L_2(\partial\omega_j^\varepsilon)} = \sum_{k \in \mathbb{Z}} \frac{|\partial\omega_k|}{\pi(b+1)R_2} (a\tilde{u}^0, v^\varepsilon)_{L_2(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon))} \\ &- \varepsilon \sum_{k \in \mathbb{Z}} (a\tilde{u}^0 \nabla X_k^\varepsilon, \nabla v^\varepsilon)_{L_2(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)} - \varepsilon \sum_{k \in \mathbb{Z}} (\nabla a\tilde{u}^0, v^\varepsilon \nabla X_k^\varepsilon)_{L_2(B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}, \end{aligned}$$

and by (2.12)

$$\begin{aligned} \left| (a\tilde{u}^0, v^\varepsilon)_{L_2(\partial\theta^\varepsilon)} - \sum_{k \in \mathbb{Z}} \frac{|\partial\omega_k|}{\pi(b+1)R_2} (a\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon))} \right| \\ \leq C \|\tilde{u}^0\|_{W_2^1(\{x: |\tau| < bR_2\varepsilon\})} \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}. \end{aligned} \quad (5.17)$$

Let  $\xi = (\xi_1, \xi_2)$  be Cartesian coordinates in  $\mathbb{R}^2$ ,  $\Xi$  be a square  $\Xi := \{\xi : |\xi_1| < bR_2, |\xi_2| < bR_2\} \subset \mathbb{R}^2$ . Consider the boundary value problem

$$\begin{aligned} \Delta \tilde{X} &= 0 \quad \text{in} \quad \Xi \setminus B_{\frac{b+1}{2}R_2}(0), \quad \frac{\partial \tilde{X}}{\partial \nu} = 1 \quad \text{on} \quad \partial B_{\frac{b+1}{2}R_2}(0), \\ \frac{\partial \tilde{X}}{\partial \nu} &= -\frac{\pi(b+1)}{2b} \quad \text{on} \quad \{\xi : |\xi_1| < bR_2, \xi_2 = -bR_2\}, \\ \frac{\partial \tilde{X}}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Xi \setminus \{\xi : |\xi_1| < bR_2, \xi_2 = -bR_2\}. \end{aligned}$$

The solvability condition of this problem is satisfied and it is solvable in  $C^1(\overline{\Xi \setminus B_{\frac{b+1}{2}R_2}(0)})$ .

To prove the latter, it is sufficient to seek the solution as  $\tilde{X} = -(\xi_2 - bR_2)^2/(4bR_2) + \hat{X}$ .

In a vicinity of each point  $y_k^\varepsilon$  we introduce rescaled variables  $\xi^j = (\xi_1^j, \xi_2^j)$  by the rule  $x = \varepsilon\eta(\xi_1^j \rho'(\varepsilon s_k) - \xi_2^j \nu^0(\varepsilon s_k)) + y_k^\varepsilon$ . The axes  $\xi_2^j = 0$  and  $\xi_1^j = 0$  are directed along the tangential and normal vectors  $\rho'(\varepsilon s_k)$  and  $\nu^0(\varepsilon s_k)$  to the curve  $\gamma$  at the point  $\varepsilon s_k$ , and the point  $\xi^j = 0$  is located at  $y_k^\varepsilon$ . We define a sequence of squares  $\Xi_j^\varepsilon := \{x : \xi^j \in \Xi\}$ . Since the curve  $\gamma$  is  $C^2$ -smooth and has the bounded curvature, due to (2.1) the tangential vectors  $\rho'(\varepsilon s_k)$  obey the uniform in small  $\varepsilon$  and all  $k \in \mathbb{Z}$  estimate

$$|\rho'(\varepsilon s_k) - \rho'(\varepsilon s_{k+1})| \leq \int_{\varepsilon s_k}^{\varepsilon s_{k+1}} |\rho''(t)| dt \leq C\varepsilon.$$

Hence, by (A2) for sufficiently small  $\varepsilon$  the squares  $\Xi_j^\varepsilon$  do not intersect with  $\omega_i^\varepsilon$  for all  $i \neq j$ . Writing the integral identity for  $\tilde{X}(\xi^j)$  and proceeding as in the proof of (5.17), one gets

$$\begin{aligned} \left| \sum_{k \in \mathbb{Z}} \frac{|\partial\omega_k|}{\pi(b+1)R_2} (a\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{\frac{b+1}{2}R_2\varepsilon\eta}(y_k^\varepsilon))} - \sum_{k \in \mathbb{Z}} \frac{|\partial\omega_k|}{2bR_2} (a\tilde{u}^0, v^\varepsilon)_{L_2(\Upsilon_j^\varepsilon)} \right| \\ \leq C \|\tilde{u}^0\|_{W_2^1(\{x: |\tau| < bR_2\varepsilon\})} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}, \end{aligned} \quad (5.18)$$

$$\Upsilon_j^\varepsilon := \{x : |\xi_1^j| < bR_2\varepsilon, \xi_2^j = -bR_2\}.$$

Together with (5.17) it yields

$$\begin{aligned} \left| (a\tilde{u}^0, v^\varepsilon)_{L_2(\partial\theta^\varepsilon)} - \sum_{k \in \mathbb{Z}} \frac{|\partial\omega_k|}{2bR_2} (a\tilde{u}^0, v^\varepsilon)_{L_2(\Upsilon_j^\varepsilon)} \right| \\ \leq C \|\tilde{u}^0\|_{W_2^1(\{x: |\tau| < bR_2\varepsilon\})} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned} \quad (5.19)$$

Let us estimate the difference

$$\sum_{k \in \mathbb{Z}} \frac{|\partial \omega_k|}{2bR_2} (a\tilde{u}^0, v^\varepsilon)_{L_2(\Upsilon_j^\varepsilon)} - (\alpha a\tilde{u}^0, v^\varepsilon)_{L_2(\tilde{\gamma})}.$$

For each  $|\xi_1^k| < bR_2$  by  $\tau_k^\varepsilon(\xi_1^k)$  we denote the value of the variable  $\tau$  corresponding to the point  $x = \varepsilon\eta\rho'(\varepsilon s_k) - bR_2\varepsilon\eta\nu^0(\varepsilon s_k)$ . It is easy to check that

$$|\tau_j^\varepsilon(\xi_1^j) - bR_2\varepsilon\eta| \leq C\varepsilon^2 \quad (5.20)$$

uniformly in  $k \in \mathbb{Z}$ , sufficiently small  $\varepsilon$ , and  $|\xi_1^k| < bR_2$ . We also observe that the integration over  $\tilde{\gamma}$  can be expressed as the integration w.r.t.  $s \in \mathbb{R}$  with the differential  $(1 - (b+1)R_2\varepsilon K(s)) ds$ , where the function  $K(s)$  was introduced in the proof of Lemma 5.2. Integrating by parts, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \frac{|\partial \omega_k|}{2bR_2} (a\tilde{u}^0, v^\varepsilon)_{L_2(\Upsilon_j^\varepsilon)} &= \sum_{k \in \mathbb{Z}} \frac{|\partial \omega_k|}{2bR_2} \int_{s_k\varepsilon - bR_2\varepsilon\eta}^{s_k\varepsilon + bR_2\varepsilon\eta} (a\tilde{u}^0 \overline{v^\varepsilon})|_{\tau = -(b+1)R_2\varepsilon} ds \\ &\quad + \sum_{k \in \mathbb{Z}} \frac{|\partial \omega_k|}{2bR_2} \int_{s_k\varepsilon - bR_2\varepsilon\eta}^{s_k\varepsilon + bR_2\varepsilon\eta} ds \int_{-(b+1)R_2\varepsilon}^{\tau_j^\varepsilon\left(\frac{s-s_k\varepsilon}{\varepsilon\eta}\right)} \frac{\partial}{\partial \tau} (a\tilde{u}^0 \overline{v^\varepsilon}) d\tau. \end{aligned}$$

By (5.20) and Lemma 3.1 it implies

$$\begin{aligned} \left| \sum_{k \in \mathbb{Z}} \frac{|\partial \omega_k|}{2bR_2} \left( (a\tilde{u}^0, v^\varepsilon)_{L_2(\Upsilon_j^\varepsilon)} - \int_{s_k\varepsilon - bR_2\varepsilon\eta}^{s_k\varepsilon + bR_2\varepsilon\eta} (a\tilde{u}^0 \overline{v^\varepsilon})|_{\tau = -(b+1)R_2\varepsilon} ds \right) \right| \\ \leq C \|\tilde{u}^0\|_{W_2^1(\{x: |\tau| < (b+1)R_2\varepsilon\})} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned}$$

This estimate, the definition of the function  $\alpha_\varepsilon$ , and (3.4) yield

$$\begin{aligned} &\left| \sum_{k \in \mathbb{Z}} \frac{|\partial \omega_k|}{2bR_2} \left( (a\tilde{u}^0, v^\varepsilon)_{L_2(\Upsilon_j^\varepsilon)} - (\alpha a\tilde{u}^0, v^\varepsilon)_{L_2(\tilde{\gamma})} \right) \right| \\ &\leq \left| \int_{\mathbb{R}} (\alpha - \alpha_\varepsilon) (a\tilde{u}^0 \overline{v^\varepsilon})|_{\tau = -(b+1)R_2\varepsilon} ds \right| + C \left( \varepsilon \|\tilde{u}^0\|_{W_2^1(\Omega)} + \|\tilde{u}^0\|_{W_2^1(\{x: |\tau| < (b+1)R_2\varepsilon\})} \right) \\ &\leq \left( \left( \int_{\mathbb{R}} |\alpha - \alpha_\varepsilon|^2 |\tilde{u}^0|_{\tau = -(b+1)R_2\varepsilon}^2 ds \right)^{1/2} \right. \\ &\quad \left. + C \left( \varepsilon \|\tilde{u}^0\|_{W_2^1(\Omega)} + \|\tilde{u}^0\|_{W_2^1(\{x: |\tau| < (b+1)R_2\varepsilon\})} \right) \right) \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned} \quad (5.21)$$

Let  $\chi_3 = \chi_3(t)$  be an infinitely differentiable cut-off function with values in  $[0, 1]$ , being one as  $t \in [1/2, 1]$ , vanishing as  $t \in (-\infty, 0] \cup [3/2, +\infty)$ , and so that  $\chi_3(t) + \chi_3(t+1) = 1$  for  $t \in [0, 1/2]$ . Then the functions  $\chi_3(s-n)$  is the partition of the unity on  $\gamma$ , i.e.,  $\sum_{n \in \mathbb{Z}} \chi_3(s-n) = 1$  for all  $s \in \mathbb{R}$ . Employing this partition, we rewrite the first term in the right hand side of (5.21) as

$$\begin{aligned} \int_{\mathbb{R}} |\alpha - \alpha_\varepsilon|^2 |\tilde{u}^0|_{\tau = -(b+1)R_2\varepsilon}^2 ds &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |\alpha - \alpha_\varepsilon|^2 \chi_3(s-n) |\tilde{u}^0|_{\tau = -(b+1)R_2\varepsilon}^2 ds \\ &= \sum_{n \in \mathbb{Z}} \int_n^{n+\frac{3}{2}} |\alpha - \alpha_\varepsilon|^2 \chi_3(s-n) |\tilde{u}^0|_{\tau = -(b+1)R_2\varepsilon}^2 ds. \end{aligned}$$



We then integrate by parts,

$$\begin{aligned} & \int_n^{n+\frac{3}{2}} |\alpha - \alpha_\varepsilon|^2 \chi_3(s-n) |\tilde{u}^0|_{\tau=-(b+1)R_2\varepsilon}|^2 ds \\ &= - \int_n^{n+\frac{3}{2}} \left( \int_n^s |\alpha(t) - \alpha_\varepsilon(t)|^2 dt \right) \frac{d}{ds} \chi_3(s-n) |\tilde{u}^0|_{\tau=-(b+1)R_2\varepsilon}|^2 ds, \end{aligned}$$

and employ (2.11) together with Lemma 5.1,

$$\begin{aligned} & \left| \int_n^{n+\frac{3}{2}} \left( \int_n^s |\alpha(t) - \alpha_\varepsilon(t)|^2 dt \right) \frac{d}{ds} \chi_3(s-n) |\tilde{u}^0|_{\tau=-(b+1)R_2\varepsilon}|^2 ds \right| \\ & \leq C \|\tilde{u}^0|_{\tau=-(b+1)R_2\varepsilon}\|_{W_2^1(n,n+2)}^2 \int_n^{n+2} |\alpha(t) - \alpha_\varepsilon(t)|^2 dt \leq C \varkappa(\varepsilon) \|\tilde{u}^0|_{\tau=-(b+1)R_2\varepsilon}\|_{W_2^1(n,n+2)}^2, \end{aligned}$$

where  $C$  is a constant independent of  $\varepsilon$ ,  $n$ , and  $\tilde{u}^0$ . Hence, by (5.11), (3.4),

$$\begin{aligned} & \left( \int_{\mathbb{R}} |\alpha - \alpha_\varepsilon|^2 |\tilde{u}^0|_{\tau=-(b+1)R_2\varepsilon}|^2 ds \right)^{1/2} \leq C \varkappa^{1/2}(\varepsilon) \|\tilde{u}^0|_{\tau=-(b+1)R_2\varepsilon}\|_{W_2^1(\tilde{\gamma})} \\ & \leq C \varkappa^{1/2}(\varepsilon) \|f\|_{L_2(\Omega)}. \end{aligned}$$

This estimate, (5.21), (5.19) yield

$$\begin{aligned} & |(a\tilde{u}^0, v^\varepsilon)_{L_2(\partial\theta^\varepsilon)} - (\alpha a\tilde{u}^0, v^\varepsilon)_{L_2(\tilde{\gamma})}| \\ & \leq C \left( \|\tilde{u}^0\|_{W_2^1(\{x: |\tau| < (b+1)R_2\varepsilon\})} + \varkappa^{1/2}(\varepsilon) \|f\|_{L_2(\Omega)} + \varepsilon \|\tilde{u}^0\|_{W_2^1(\Omega)} \right). \end{aligned} \quad (5.22)$$

We apply Lemma 3.2 with  $v = \tilde{u}^0$ ,  $v = \frac{\partial \tilde{u}^0}{\partial x_i}$  and use (5.11) to get

$$|(a\tilde{u}^0, v^\varepsilon)_{L_2(\partial\theta^\varepsilon)} - (\alpha a\tilde{u}^0, v^\varepsilon)_{L_2(\tilde{\gamma})}| \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon)) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}.$$

We substitute this estimate and (5.16) into (5.15) and employ (3.7),

$$\|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon)) \|f\|_{L_2(\Omega)}.$$

Combining this estimate with (5.12), we complete the proof.

## 6 Dirichlet condition on boundaries of holes: delta-interaction

This section is devoted to the proof of Theorem 2.2. As in the proof of Theorem 2.4, we introduce the operator  $\tilde{\mathcal{H}}_\alpha^0$  with  $\beta$  defined in the statement of the theorem. Given  $f \in L_2(\Omega)$ , we let  $u^\varepsilon := (\mathcal{H}^\varepsilon - i)^{-1}f$ ,  $u^0 := (\mathcal{H}_\beta^0 - i)^{-1}f$ ,  $\tilde{u}^0 := (\tilde{\mathcal{H}}_\beta^0 - i)^{-1}f$ .

We begin with auxiliary lemmata.

**Lemma 6.1.** *Let  $u \in W_2^2(\Omega \setminus (\gamma \cup \tilde{\gamma})) \cap W_2^1(\Omega)$ . Then the uniform in  $\varepsilon$ ,  $k$ , and  $u$  estimate*

$$\left( \sum_{k \in \mathbb{Z}} \|u\|_{C(\overline{B_{R_2\varepsilon}(y_\varepsilon^k)})}^2 \right)^{1/2} \leq C \varepsilon^{-1/2} \|u\|_{W_2^2(\Omega \setminus (\gamma \cup \tilde{\gamma}))}$$

*holds true.*

*Proof.* We first observe that by the standard embedding theorems the function  $u$  is continuous on  $\overline{\Omega_+}$  and each term in the left hand side of the desired estimate is therefore well-defined.

We introduce the rescaled variables  $\xi^k = (\xi_1^k, \xi_2^k) = (x - y_\varepsilon^k)\varepsilon^{-1}$ . The function  $u(\varepsilon\xi^k)$  belongs to  $W_2^2(B_{R_2}(0))$  and due to the embedding  $C(\overline{B_{R_2}(0)}) \subset W_2^2(B_{R_2}(0))$  we have

$$\|u(\varepsilon\cdot)\|_{C(\overline{B_{R_2}(0)})} \leq C\|u(\varepsilon\cdot)\|_{W_2^2(B_{R_2}(0))},$$

where  $C$  is a constant independent of  $\varepsilon$  and  $u$ . Returning back to the variable  $x$ , we rewrite the latter inequality as

$$\|u\|_{C(\overline{B_{R_2\varepsilon}(y_k^\varepsilon)})}^2 \leq C \left( \|\nabla u\|_{W_2^1(B_{R_2\varepsilon}(y_k^\varepsilon))}^2 + \varepsilon^{-2} \|u\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))}^2 \right).$$

By (3.3) with  $\eta = 1$  it yields

$$\begin{aligned} \sum_{k \in \mathbb{M}_0} \|u\|_{C(\overline{B_{R_2\varepsilon}(y_k^\varepsilon)})}^2 &\leq C \left( \|\nabla u\|_{W_2^1(\tilde{\Omega}_+)}^2 + \varepsilon^{-2} \sum_{k \in \mathbb{Z}} \|u\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))}^2 \right) \\ &\leq C\varepsilon^{-1} \|u\|_{W_2^2(\Omega \setminus (\gamma \cup \tilde{\gamma}))}^2. \end{aligned}$$

□

We let

$$A(x) = \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{12}(x) & A_{22}(x) \end{pmatrix}, \quad A_k^\varepsilon := A(y_k^\varepsilon),$$

and for each  $k \in \mathbb{M}_0$  by  $Q_k^\varepsilon$  we denote the square root of the inverse matrix  $(A_k^\varepsilon)^{-1}$ , i.e.,

$$(Q_k^\varepsilon)^t (Q_k^\varepsilon) = (A_k^\varepsilon)^{-1}, \quad (Q_k^\varepsilon)^t A_k^\varepsilon (Q_k^\varepsilon) = E, \quad (6.1)$$

where  $E$  is the unit matrix. Due to condition (2.2) the matrix  $A$  is symmetric, lower-semibounded and bounded uniformly in  $x \in \overline{\Omega}$ , and this is why the matrix  $Q_k^\varepsilon$  is well-defined, lower-semibounded and bounded uniformly in  $k$  and  $\varepsilon$ . Hence, we have the estimate

$$0 < C|z| \leq |Q_k^\varepsilon z| \leq C^{-1}|z| \quad (6.2)$$

that is uniform in  $k \in \mathbb{M}_0$ ,  $z \in \mathbb{R}^2$ , and sufficiently small  $\varepsilon$ .

By assumption (A12)  $\mu(\varepsilon) \rightarrow +0$  as  $\varepsilon \rightarrow +0$  and

$$\eta(\varepsilon) = e^{-\frac{1}{\varepsilon(K+\mu(\varepsilon))}}. \quad (6.3)$$

For each  $k \in \mathbb{M}_0$  we introduce the rescaled variables  $\xi^k = (\xi_1^k, \xi_2^k) = (x - y_k^\varepsilon)\varepsilon^{-1}$  and define the ellipses  $\tilde{B}_R^k := \{x : |Q_k^\varepsilon \xi^k| < R\}$ . In view of (6.2), (6.3) there exists an absolute positive constant  $R_4$  such that for all sufficiently small  $\varepsilon$  and  $k \in \mathbb{M}_0$

$$\omega_k^\varepsilon \subseteq \tilde{B}_{R_4\eta}^k \subset \tilde{B}_{R_4}^k \subseteq B_{\frac{R_2}{2}\varepsilon}(y_k^\varepsilon). \quad (6.4)$$

Hence, the areas of the ellipses  $\tilde{B}_{R_4}^k$  are bounded uniformly in  $k \in \mathbb{M}_0$  and  $\varepsilon$ ,

$$|\tilde{B}_{R_4}^k| \leq C\varepsilon. \quad (6.5)$$

We define the function

$$W^\varepsilon(x) := \begin{cases} -\frac{1}{\ln \eta(\varepsilon)} \ln \frac{|Q_k^\varepsilon \xi^k|}{R_4\eta(\varepsilon)}, & x \in \tilde{B}_{R_4}^k \setminus \tilde{B}_{R_4\eta}^k, \quad k \in \mathbb{M}_0, \\ 0, & x \in \tilde{B}_{R_4\eta}^k, \quad k \in \mathbb{M}_0, \\ 1, & \text{otherwise.} \end{cases} \quad (6.6)$$

It is clear that it is infinitely differentiable everywhere in  $\overline{\Omega}$  except the boundaries  $\partial\tilde{B}_{R_4}^k$  and  $\partial\tilde{B}_{R_4\eta}^k$ ,  $k \in \mathbb{M}_0$ , and continuous everywhere in  $\overline{\Omega}$ . The function  $W^\varepsilon$  is uniformly bounded in  $\overline{\Omega}$  obeying the estimate  $0 \leq W^\varepsilon \leq 1$ . An important property of this function is that it vanishes on  $\partial\tilde{B}_{R_4\eta}^k$ ,  $k \in \mathbb{M}_0$ , and thus on  $\overline{\theta_0^\varepsilon}$ . Hence, the latter is true also for  $W^\varepsilon(x)\tilde{u}^0(x)$ . We then denote  $v^\varepsilon := u^\varepsilon - W^\varepsilon\tilde{u}^0 \in W_2^1(\Omega^\varepsilon)$ , and write the boundary value problem for this function,

$$\begin{aligned} & \left( - \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^2 A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 - i \right) v^\varepsilon = (1 - W^\varepsilon)f - f_1 \\ & - 2 \sum_{i,j=1}^2 A_{ij} \frac{\partial W^\varepsilon}{\partial x_i} \frac{\partial \tilde{u}^0}{\partial x_j} \quad \text{in } \Omega^\varepsilon \setminus \bigcup_{k \in \mathbb{M}_0} \partial\tilde{B}_{R_4}^k \cup \partial\tilde{B}_{R_4\eta}^k, \\ & f_1 := \tilde{u}^0 \left( - \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^2 A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} \right) W^\varepsilon, \\ & v^\varepsilon = 0 \quad \text{on } \partial\Omega \cup \partial\theta_0^\varepsilon, \quad \left( \frac{\partial}{\partial N^\varepsilon} + a \right) v^\varepsilon = - \left( \frac{\partial}{\partial N^\varepsilon} + a \right) \tilde{u}^0 \quad \text{on } \partial\theta_1^\varepsilon, \\ & [v^\varepsilon]_{\tilde{\gamma}} = 0, \quad \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right]_{\tilde{\gamma}} + \beta \tilde{u}^0|_{\tilde{\gamma}} = 0, \quad [v^\varepsilon]_{\partial\tilde{B}_{R_4}^k} = 0, \quad [v^\varepsilon]_{\partial\tilde{B}_{R_4\eta}^k} = 0, \\ & \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right] = -\tilde{u}^0 \left[ \frac{\partial W^\varepsilon}{\partial \tilde{N}^\varepsilon} \right] \quad \text{on } \partial\tilde{B}_{R_4}^k, \quad \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right] = -\tilde{u}^0 \left[ \frac{\partial W^\varepsilon}{\partial \tilde{N}^\varepsilon} \right] \quad \text{on } \partial\tilde{B}_{R_4\eta}^k. \end{aligned} \tag{6.7}$$

Here  $k \in \mathbb{M}_0$ , as in (2.6), (5.10),  $[\cdot]$  denotes the jump of a function,

$$[u]_{\partial\tilde{B}_R^k} = u|_{|Q_k^\varepsilon \xi^k|=R+0} - u|_{|Q_k^\varepsilon \xi^k|=R-0} \quad \text{and} \quad \frac{\partial}{\partial \tilde{N}^\varepsilon} := \sum_{i,j=1}^2 A_{ij} \tilde{\nu}_i^\varepsilon \frac{\partial}{\partial x_j},$$

where  $\tilde{\nu}^\varepsilon = (\tilde{\nu}_1^\varepsilon, \tilde{\nu}_2^\varepsilon)$  is the inward normal to  $\partial\tilde{B}_{R_4}^k$  and  $\partial\tilde{B}_{R_4\eta}^k$ . By straightforward calculations we check that on  $\partial\tilde{B}_{R_4}^k$

$$\begin{aligned} \tilde{\nu}^\varepsilon &= - \frac{(\overline{A_k^\varepsilon})^{-1} \xi^k}{|(\overline{A_k^\varepsilon})^{-1} \xi^k|}, \quad \nabla_x \ln \frac{|Q_k^\varepsilon \xi^k|}{R_4 \eta} = \frac{(\overline{A_k^\varepsilon})^{-1} \xi^k}{\varepsilon |Q_k^\varepsilon \xi^k|^2} = \frac{(\overline{A_k^\varepsilon})^{-1} \xi^k}{\varepsilon R_4^2}, \\ - \left[ \frac{\partial W^\varepsilon}{\partial \tilde{N}^\varepsilon} \right] &= \frac{1}{\ln \eta} \left( \nu^\varepsilon, A \nabla_x \ln \frac{|Q_k^\varepsilon \xi^k|}{R_4 \eta} \right)_{\mathbb{R}^2} = - \frac{((A_k^\varepsilon)^{-1} \xi^k, A (A_k^\varepsilon)^{-1} \xi^k)_{\mathbb{R}^2}}{|(A_k^\varepsilon)^{-1} \xi^k| R_4^2 \varepsilon \ln \eta}. \end{aligned}$$

Since on  $\partial\tilde{B}_{R_4}^k$

$$|A_{ij}(x) - A_{ij}(y_k^\varepsilon)| \leq C\varepsilon, \quad ((A_k^\varepsilon)^{-1} \xi^k, \xi^k)_{\mathbb{R}^2} = |Q_k^\varepsilon \xi^k|^2 = R_4^2,$$

we get

$$\left| \left[ \frac{\partial W^\varepsilon}{\partial \tilde{N}^\varepsilon} \right]_{\partial\tilde{B}_{R_4}^k} - q_k^\varepsilon \right| \leq \frac{C}{|\ln \eta|}, \quad q_k^\varepsilon := \frac{1}{|(A_k^\varepsilon)^{-1} \xi^k| \varepsilon \ln \eta} + \sum_{j=1}^2 \overline{A_j} \tilde{\nu}_j^\varepsilon, \tag{6.8}$$

uniformly in  $x \in \partial\tilde{B}_{R_4}^k$ ,  $k \in \mathbb{M}_0$ , and sufficiently small  $\varepsilon$ . In the same way we obtain

$$\left| \left[ \frac{\partial W^\varepsilon}{\partial \tilde{N}^\varepsilon} \right]_{\partial\tilde{B}_{R_4\eta}^k} \right| \leq \frac{C}{\varepsilon \eta |\ln \eta|}. \tag{6.9}$$

The integral identity associated with problem (6.7) reads as

$$\mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon) - i \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 = - \left( \frac{\partial \tilde{u}^0}{\partial N^\varepsilon}, v^\varepsilon \right)_{L_2(\partial\theta_1^\varepsilon)} - (a \tilde{u}^0, v^\varepsilon)_{L_2(\partial\theta_1^\varepsilon)} + (\beta \tilde{u}^0, v^\varepsilon)_{L_2(\tilde{\gamma})}$$

$$\begin{aligned}
& + \sum_{k \in \mathbb{M}_0} \left( \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right], v^\varepsilon \right)_{L_2(\partial \tilde{B}_{R_4}^k)} + \sum_{k \in \mathbb{M}_0} \left( \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right], v^\varepsilon \right)_{L_2(\partial \tilde{B}_{R_4 \eta}^k)} \\
& + ((1 - W^\varepsilon)f, v^\varepsilon)_{L_2(\Omega^\varepsilon)} + (f_1, v^\varepsilon)_{L_2(\Omega^\varepsilon)} \\
& - 2 \sum_{k \in \mathbb{M}_0} \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial W^\varepsilon}{\partial x_i} \frac{\partial \tilde{u}^0}{\partial x_j}, v^\varepsilon \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)}.
\end{aligned}$$

We integrate by parts in the last term employing the definition of  $W^\varepsilon$ ,

$$\begin{aligned}
& \sum_{k \in \mathbb{M}_0} \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial W^\varepsilon}{\partial x_i} \frac{\partial \tilde{u}^0}{\partial x_j}, v^\varepsilon \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} = \sum_{k \in \mathbb{M}_0} \sum_{i,j=1}^2 \left( A_{ij} \frac{\partial(W^\varepsilon - 1)}{\partial x_i} \frac{\partial \tilde{u}^0}{\partial x_j}, v^\varepsilon \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} \\
& = \left( (1 - W^\varepsilon) \frac{\partial}{\partial x_i} A_{ij} \frac{\partial \tilde{u}^0}{\partial x_j}, v^\varepsilon \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} - \left( (1 - W^\varepsilon) A_{ij} \frac{\partial \tilde{u}^0}{\partial x_j}, \frac{\partial v^\varepsilon}{\partial x_i} \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)}.
\end{aligned}$$

This identity together with the previous one and the boundary conditions in (6.7) yield

$$\begin{aligned}
& \mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon) - \mathfrak{i} \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 = - \left( \frac{\partial \tilde{u}^0}{\partial \tilde{N}^\varepsilon}, v^\varepsilon \right)_{L_2(\partial \theta_1^\varepsilon)} - (a \tilde{u}^0, v^\varepsilon)_{L_2(\partial \theta_1^\varepsilon)} \\
& + (\alpha \tilde{u}^0, v^\varepsilon)_{L_2(\tilde{\gamma})} + \sum_{k \in \mathbb{M}_0} \left( \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right], v^\varepsilon \right)_{L_2(\partial \tilde{B}_{R_4}^k)} + \sum_{k \in \mathbb{M}_0} \left( \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right], v^\varepsilon \right)_{L_2(\partial \tilde{B}_{R_4 \eta}^k)} \\
& + ((1 - W^\varepsilon)f, v^\varepsilon)_{L_2(\Omega^\varepsilon)} + 2 \left( (1 - W^\varepsilon) A_{ij} \frac{\partial \tilde{u}^0}{\partial x_j}, \frac{\partial v^\varepsilon}{\partial x_i} \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} \\
& + (f_1, v^\varepsilon)_{L_2(\Omega^\varepsilon)} - 2 \left( (1 - W^\varepsilon) \frac{\partial}{\partial x_i} A_{ij} \frac{\partial \tilde{u}^0}{\partial x_j}, v^\varepsilon \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)}.
\end{aligned} \tag{6.10}$$

Let us estimate the right hand side of the obtained identity. We first extend  $v^\varepsilon$  by zero inside  $\theta_0^\varepsilon$ . Since  $v^\varepsilon$  vanishes on  $\partial \theta_0^\varepsilon$ , the extension still belongs to  $\Omega \setminus \theta_1^\varepsilon$  and has the same  $L_2$ - and  $W_2^1$ -norms.

By Lemma 3.3 we have

$$\left| \left( \frac{\partial \tilde{u}^0}{\partial \tilde{N}^\varepsilon}, v^\varepsilon \right)_{L_2(\partial \theta_1^\varepsilon)} + (a \tilde{u}^0, v^\varepsilon)_{L_2(\partial \theta_1^\varepsilon)} \right| \leq C \varepsilon^{1/2} \eta |\ln \eta|^{1/2} \|\tilde{u}^0\|_{W_2^2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \tag{6.11}$$

It follows from (6.1) that

$$\sum_{i,j=1}^2 A_{ij}(y_k^\varepsilon) \frac{\partial^2 W^\varepsilon}{\partial x_i \partial x_j} = 0 \quad \text{in} \quad \tilde{B}_{R_4}^k \setminus \tilde{B}_{R_4 \eta}^k, \quad k \in \mathbb{M}_0.$$

This identity yields an estimate for  $f_1$ ,

$$|f_1| \leq \frac{C |\tilde{u}^0|}{|\ln \eta| |x - y_k^\varepsilon|} \quad \text{in} \quad \tilde{B}_{R_4}^k \setminus \tilde{B}_{R_4 \eta}^k, \quad k \in \mathbb{M}_0.$$

Employing this estimate, (6.3), Lemma 3.2 with  $\eta = 1$ , Lemma 6.1, and Cauchy-Schwarz

inequality, we can estimate two other terms in the right hand side of (6.10),

$$\begin{aligned}
& |((1 - W^\varepsilon)f, v^\varepsilon)_{L_2(\Omega^\varepsilon)} + (f_1, v^\varepsilon)_{L_2(\Omega^\varepsilon)}| \\
& \leq \sum_{k \in \mathbb{M}_0} |((1 - W^\varepsilon)f, v^\varepsilon)_{L_2(\Omega^\varepsilon)} + (f_1, v^\varepsilon)_{L_2(\Omega^\varepsilon)}| \\
& \leq C \left( \sum_{k \in \mathbb{M}_0} \|f\|_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} \|v^\varepsilon\|_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} \right. \\
& \quad \left. + \sum_{k \in \mathbb{M}_0} \frac{\|\tilde{u}^0\|_{C(\overline{B}_{R_4}^k)}}{|\ln \eta|} \left\| \frac{1}{|\cdot - y_k^\varepsilon|} \right\|_{L_2(B_{R_2^\varepsilon}(y_k^\varepsilon) \setminus \tilde{B}_{R_4}^k)} \|v^\varepsilon\|_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} \right) \\
& \leq C \left( \sum_{k \in \mathbb{M}_0} \left( \|f\|_{L_2(\tilde{B}_{R_4}^k)}^2 + \frac{1}{|\ln \eta|} \|\tilde{u}^0\|_{C(\overline{B}_{R_2^\varepsilon}(y_k^\varepsilon))}^2 \right) \right)^{1/2} \\
& \quad \left( \sum_{k \in \mathbb{M}_0} \|v^\varepsilon\|_{L_2(B_{R_2^\varepsilon}(y_k^\varepsilon) \setminus \omega_k^\varepsilon)}^2 \right)^{1/2} \\
& \leq C\varepsilon^{1/2} (\|f\|_{L_2(\Omega)} + \|\tilde{u}^0\|_{W_2^2(\Omega)}) \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}.
\end{aligned} \tag{6.12}$$

In the same way we get

$$\begin{aligned}
& \left| -2 \left( (1 - W^\varepsilon) \frac{\partial}{\partial x_i} A_{ij} \frac{\partial \tilde{u}^0}{\partial x_j}, v^\varepsilon \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} + 2 \left( (1 - W^\varepsilon) A_{ij} \frac{\partial \tilde{u}^0}{\partial x_j}, \frac{\partial v^\varepsilon}{\partial x_i} \right)_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} \right| \\
& \leq C\varepsilon^{1/2} \|\tilde{u}^0\|_{W_2^2(\Omega \setminus \tilde{\gamma})} \|v^\varepsilon\|_{W_2^1(\Omega)}.
\end{aligned} \tag{6.13}$$

We need extra auxiliary lemmata.

**Lemma 6.2.** *For all  $v \in W_2^1(\Omega^\varepsilon, \partial\theta_0^\varepsilon)$  the uniform in  $k \in \mathbb{M}_0$ ,  $\varepsilon$ , and  $v$  estimates*

$$\begin{aligned}
& \|v\|_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} \leq C\varepsilon\eta \|\nabla v\|_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)}, \\
& \|v\|_{L_2(\partial\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)} \leq C\varepsilon^{1/2}\eta^{1/2} \|\nabla v\|_{L_2(\tilde{B}_{R_4}^k \setminus \omega_k^\varepsilon)}, \\
& \left( \sum_{k \in \mathbb{M}_0} \|v\|_{L_2(\partial\tilde{B}_{R_4}^k)}^2 \right)^{1/2} \leq C\|v\|_{L_2(\Omega^\varepsilon)}
\end{aligned}$$

hold true.

The proof of this lemma consists just in the integration of (4.7) over  $\tilde{B}_{R_4}^k$ ,  $\partial\tilde{B}_{R_4}^k$ , and  $\partial\tilde{B}_{R_4}^k$ .

By (6.9), (6.3), Lemmata 3.1, 6.1, 6.2 we have

$$\begin{aligned}
& \left| \sum_{k \in \mathbb{M}_0} \left( \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right], v^\varepsilon \right)_{L_2(\partial\tilde{B}_{R_4}^k)} \right| \leq \frac{C}{\varepsilon^{1/2}\eta^{1/2}|\ln \eta|} \sum_{k \in \mathbb{M}_0} \|\tilde{u}^0\|_{C(\overline{B}_{R_4}^k)} \|v^\varepsilon\|_{L_2(\partial\tilde{B}_{R_4}^k)} \\
& \leq C\varepsilon^{1/2} \|\tilde{u}^0\|_{W_2^2(\Omega \setminus \tilde{\gamma})} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}.
\end{aligned}$$

We employ (6.4), (6.8), (5.11), Lemma 6.1, and Lemma 3.2 with  $\eta = 1$  to simplify one of the remaining boundary terms in the right hand side of (6.10),

$$\begin{aligned}
& \left| \sum_{k \in \mathbb{M}_0} \left( \left[ \frac{\partial v^\varepsilon}{\partial \tilde{N}^\varepsilon} \right], v^\varepsilon \right)_{L_2(\partial\tilde{B}_{R_4}^k)} - \sum_{k \in \mathbb{M}_0} (q_k^\varepsilon \tilde{u}^0, v^\varepsilon)_{L_2(\partial\tilde{B}_{R_4}^k)} \right| \\
& \leq \frac{C}{\varepsilon^{1/2}|\ln \eta|} \|\tilde{u}^0\|_{W_2^2(\Omega \setminus \tilde{\gamma})} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \leq C\varepsilon^{1/2} \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}.
\end{aligned} \tag{6.14}$$

We substitute this estimate and (5.11), (6.3), (6.11), (6.9), (6.12), (6.13) into (6.10),

$$\begin{aligned} \left| \mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon) - \mathfrak{i} \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 \right| &\leq C\varepsilon^{1/2} \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \\ &\quad + \left| (\beta \tilde{u}^0, v^\varepsilon)_{L_2(\tilde{\gamma})} - \sum_{k \in \mathbb{M}_0} (q_k^\varepsilon \tilde{u}^0, v^\varepsilon)_{L_2(\partial \tilde{B}_{R_4}^k)} \right|. \end{aligned}$$

It remains to estimate the last two terms in the right hand side of the latter inequality.

Denote

$$\langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} := \frac{1}{\pi R^2 \varepsilon^2} \int_{B_{R_2\varepsilon}(y_k^\varepsilon)} \tilde{u}^0 dx, \quad \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} := \frac{1}{\pi R^2 \varepsilon^2} \int_{B_{R_2\varepsilon}(y_k^\varepsilon)} v^\varepsilon dx.$$

**Lemma 6.3.** *The uniform in  $\varepsilon$  and  $k \in \mathbb{M}_0$  estimates*

$$\|\tilde{u}^0 - \langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}\|_{L_2(\partial \tilde{B}_{R_4}^k)} \leq C\varepsilon \|\nabla \tilde{u}^0\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))}, \quad (6.15)$$

$$\|v^\varepsilon - \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}\|_{L_2(\partial \tilde{B}_{R_4}^k)} \leq C\varepsilon \|\nabla v^\varepsilon\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))},$$

$$\|\tilde{u}^0 - \langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}\|_{L_2(\partial B_{R_2\varepsilon}(y_k^\varepsilon))} \leq C\varepsilon \|\nabla \tilde{u}^0\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))}, \quad (6.16)$$

$$\|v^\varepsilon - \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}\|_{L_2(\partial B_{R_2\varepsilon}(y_k^\varepsilon))} \leq C\varepsilon \|\nabla v^\varepsilon\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))},$$

$$\left( \sum_{k \in \mathbb{M}_0} |\langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}|^2 \right)^{1/2} \leq C\varepsilon^{-1/2} \|\tilde{u}^0\|_{W_2^1(\Omega)}, \quad (6.17)$$

$$\left( \sum_{k \in \mathbb{M}_0} |\langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}|^2 \right)^{1/2} \leq C\varepsilon^{-1/2} \|v^\varepsilon\|_{W_2^1(\Omega)}$$

hold true.

*Proof.* To prove (6.17), it is sufficient to see that

$$|\langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}| \leq C\varepsilon^{-1} \|\tilde{u}^0\|_{L_2(\tilde{B}_{R_4}^k)}, \quad |\langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}| \leq C\varepsilon^{-1} \|v^\varepsilon\|_{L_2(\tilde{B}_{R_4}^k)}$$

and apply (3.3) with  $\eta = 1$ .

Rescaling  $B_{R_2\varepsilon}(y_k^\varepsilon)$  in  $(\varepsilon\eta)^{-1}$  times and employing Poincaré inequality [36, Ch. I, Sec. 1, Ineq. (1.5)], we get

$$\begin{aligned} \|\tilde{u}^0 - \langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))} &\leq C\varepsilon \|\nabla \tilde{u}^0\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))}, \\ \|v^\varepsilon - \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))} &\leq C\varepsilon \|\nabla v^\varepsilon\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))}. \end{aligned} \quad (6.18)$$

In view of (6.2) there exists a number  $R$  independent of  $\varepsilon$  and  $k \in \mathbb{M}_0$  such that  $B_{R\varepsilon}(y_k^\varepsilon) \subseteq \tilde{B}_{R_4}^k$  for all  $\varepsilon$  and  $k \in \mathbb{M}_0$ . Employing this fact, estimates (6.18) and integrating (3.6) with  $u = \tilde{u}^0 - \langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}$ ,  $u = v^\varepsilon - \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)}$  over  $\partial \tilde{B}_{R_4}^k$ ,  $\partial B_{R_2\varepsilon}(y_k^\varepsilon)$ , we prove (6.15), (6.16).  $\square$

As it follows from definition (6.8) of the function  $q_k^\varepsilon$ , this function is bounded uniformly in  $\varepsilon$  and  $k \in \mathbb{M}_0$ . Then Lemma 6.3 and Lemma 3.2 with  $\eta = 1$  imply

$$\begin{aligned} \left| \sum_{k \in \mathbb{M}_0} (q_k^\varepsilon \tilde{u}^0, v^\varepsilon)_{L_2(\partial \tilde{B}_{R_4}^k)} - \sum_{k \in \mathbb{M}_0} \langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} \int_{\partial \tilde{B}_{R_4}^k} q_k^\varepsilon ds \right| \\ \leq C\varepsilon \|\tilde{u}^0\|_{W_2^1(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned} \quad (6.19)$$

Let us estimate the integrals  $\int_{\partial \tilde{B}_{R_4}^k} q_k^\varepsilon ds$ .

The boundary  $\partial \tilde{B}_{R_4}^k$  is described by the equation  $|\mathbf{Q}_k^\varepsilon \xi^k| = R_4$  and we can parameterize it as

$$\xi^k = R_4(\mathbf{Q}_k^\varepsilon)^{-1}e_\varphi, \quad e_\varphi = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \quad \varphi \in [0, 2\pi).$$

Hence, due to (6.1),

$$\left| \frac{d\xi^k}{d\varphi} \right|^2 = R_4^2(e'_\varphi, \mathbf{A}_k^\varepsilon e'_\varphi)_{\mathbb{R}^2},$$

where  $'$  denotes the derivative w.r.t.  $\varphi$ . By straightforward calculations we check that

$$\begin{aligned} |(\mathbf{A}_k^\varepsilon)^{-1}\xi^k|^2 &= R_4^2|(\mathbf{Q}_k^\varepsilon)^t e_\varphi|^2 = R_4^2(e_\varphi, (\mathbf{A}_k^\varepsilon)^{-1}e_\varphi)_{\mathbb{R}^2} = \frac{R_4^2(e'_\varphi, \mathbf{A}_k^\varepsilon e'_\varphi)_{\mathbb{R}^2}}{\det \mathbf{A}_k^\varepsilon}, \\ \int_{\partial \tilde{B}_{R_4}^k} \frac{ds}{|(\mathbf{A}_k^\varepsilon)^{-1}\xi^k|} &= \varepsilon \int_0^{2\pi} d\varphi = \frac{2\pi\varepsilon}{\det \mathbf{A}_k^\varepsilon}. \end{aligned}$$

These relations and (6.17), (6.3) imply

$$\begin{aligned} \left| \sum_{k \in \mathbb{M}_0} \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} \langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} \int_{\partial \tilde{B}_{R_4}^k} ds + \sum_{k \in \mathbb{M}_0} \frac{2\pi\varepsilon(K + \mu)}{\det \mathbf{A}_k^\varepsilon} \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} \langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} \right| \\ \leq C\varepsilon^{1/2} \|\tilde{u}^0\|_{W_2^2(\Omega \setminus \tilde{\gamma})} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}, \end{aligned}$$

and by (6.16)

$$\begin{aligned} \left| \sum_{k \in \mathbb{M}_0} \frac{2\pi\varepsilon}{\det \mathbf{A}_k^\varepsilon} \langle v^\varepsilon \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} \langle \tilde{u}^0 \rangle_{B_{R_2\varepsilon}(y_k^\varepsilon)} - \sum_{k \in \mathbb{M}_0} \left( \frac{1}{R_2 \det \mathbf{A}} \tilde{u}^0, v^\varepsilon \right)_{L_2(\partial B_{R_2\varepsilon}(y_k^\varepsilon))} \right| \\ \leq C\varepsilon^{1/2} \|\tilde{u}^0\|_{W_2^2(\Omega \setminus \tilde{\gamma})} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned}$$

These two inequalities and (6.19), (6.14), (5.11) lead us to the estimate

$$\begin{aligned} \left| \mathfrak{h}^\varepsilon(v^\varepsilon, v^\varepsilon) - \mathfrak{i} \|v^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 \right| &\leq C\varepsilon^{1/2} \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \\ &+ (K + \mu) \left| \sum_{k \in \mathbb{M}_0} \frac{1}{R_2 \det \mathbf{A}_k^\varepsilon} (\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{R_2\varepsilon}(y_k^\varepsilon))} - \left( \frac{\alpha}{\det \mathbf{A}} \tilde{u}^0, v^\varepsilon \right)_{L_2(\tilde{\gamma})} \right|. \end{aligned}$$

In the same way how (5.22) was proven, one can show that

$$\begin{aligned} \left| \sum_{k \in \mathbb{M}_0} \frac{1}{R_2 \det \mathbf{A}_k^\varepsilon} (\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{R_2\varepsilon}(y_k^\varepsilon))} - \left( \frac{\alpha}{\det \mathbf{A}} \tilde{u}^0, v^\varepsilon \right)_{L_2(\tilde{\gamma})} \right| \\ \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon)(K + \mu(\varepsilon))) \|f\|_{L_2(\Omega)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}. \end{aligned}$$

Two last estimates and (3.7) yield

$$\|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(K + \mu)) \|f\|_{L_2(\Omega)}.$$

It follows from Lemmata 5.2, 6.1 that

$$\left( \sum_{k \in \mathbb{M}_0} \|u^0 - \tilde{u}^0\|_{C(\overline{B_{R_2\varepsilon}(y_k^\varepsilon)})} \right)^{1/2} \leq C \|f\|_{L_2(\Omega)}.$$

Hence, due to (6.2), (6.3), (6.4),

$$\left( \sum_{k \in \mathbb{M}_0} \|(u^0 - \tilde{u}^0) \nabla W^\varepsilon\|_{L_2(\tilde{B}_{R_4}^k)}^2 \right)^{1/2} \leq \left( \sum_{k \in \mathbb{M}_0} \|u^0 - \tilde{u}^0\|_{C(\overline{B_{R_2\varepsilon}(y_k^\varepsilon)})}^2 \|\nabla W^\varepsilon\|_{L_2(\tilde{B}_{R_4}^k)}^2 \right)^{1/2}$$



$$\leq C\varepsilon^{1/2}\|f\|_{L_2(\Omega)}.$$

It allows us to rewrite (6.5) as

$$\|u^\varepsilon - u^0 W^\varepsilon\|_{W_2^1(\Omega)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(K + \mu))\|f\|_{L_2(\Omega)}, \quad (6.20)$$

and it yields (2.9). Since  $u^0 - W^\varepsilon u^0 = 0$  outside  $\widetilde{B}_{R_4\varepsilon}^k$ ,  $k \in \mathbb{M}_0$ , by Lemma 3.2 with  $\eta = 1$ , Lemma 3.6, and (6.4) we obtain

$$\|u^0 - W^\varepsilon u^0\|_{L_2(\Omega)} \leq C\varepsilon^{1/2}\|u^0\|_{W_2^1(\Omega)} \leq C\varepsilon^{1/2}\|f\|_{L_2(\Omega)}$$

that by (6.20) implies

$$\|u^\varepsilon - u^0\|_{L_2(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon))\|f\|_{L_2(\Omega)}, \quad (6.21)$$

and therefore (2.7) holds true. To prove (2.8), now it is sufficient to employ the obvious estimate

$$\|(\mathcal{H}_\beta^0 - i)^{-1}f - (\mathcal{H}_{\beta_0}^0 - i)^{-1}f\|_{W_2^1(\Omega)} \leq C\mu\|f\|_{L_2(\Omega)}. \quad (6.22)$$

As  $K = 0$ , by Lemma 3.6, Lemma 3.2 with  $\eta = 1$ , and (6.4), (6.3), we get

$$\begin{aligned} \|\nabla(1 - W^\varepsilon)u^0\|_{L_2(\Omega)} &\leq \|W^\varepsilon \nabla u^0\|_{L_2(\Omega)} + \|u^0 \nabla W^\varepsilon\|_{L_2(\Omega)} \\ &\leq C \left( \varepsilon^{1/2}\|f\|_{L_2(\Omega)} + \left( \sum_{k \in \mathbb{M}_0} \|u^0 \nabla W^\varepsilon\|_{L_2(B_{R_2\varepsilon}(y_k^\varepsilon))}^2 \right)^{1/2} \right) \\ &\leq C(\varepsilon^{1/2} + \mu^{1/2}(\varepsilon))\|f\|_{L_2(\Omega)}. \end{aligned}$$

Thus, as  $K = 0$ , due to (6.20), (6.21),

$$\|u^\varepsilon - u^0\|_{W_2^1(\Omega^\varepsilon)} \leq C(\varepsilon^{1/2} + \varkappa^{1/2}(\varepsilon))\|f\|_{L_2(\Omega)}.$$

Together with (6.22) it follows (2.10). The proof is complete.

## 7 Spectrum

In this section we study the behavior of the spectrum of the perturbed operator. We first prove a general result, Theorem 2.5, on the convergence of the perturbed spectrum. Then the rest of the section is devoted to the study of the periodic case, namely, to the proof of Theorems 2.6, 2.7.

### 7.1 Convergence of spectrum

In  $\theta^\varepsilon$  we introduce the operator  $\mathcal{H}^\theta$  acting as  $-\Delta$  subject to Dirichlet condition; the associated form is  $(\nabla u, \nabla v)_{L_2(\theta^\varepsilon)}$  on  $\dot{W}_2^1(\theta^\varepsilon)$ . Employing minimax principle and assumption (A2), one can easily make sure that

$$\inf \sigma(\mathcal{H}^\varepsilon) \geq C\varepsilon^{-2}\eta^{-2}(\varepsilon). \quad (7.1)$$

Thus, we have the estimate

$$\|(\mathcal{H}^\theta - i)^{-1}\|_{L_2(\theta^\varepsilon) \rightarrow L_2(\theta^\varepsilon)} \leq C\varepsilon^2\eta^2(\varepsilon). \quad (7.2)$$

Assuming the hypothesis of one of Theorems 2.1, 2.3, 2.4, 2.2, by  $\mathcal{H}_*^0$  we denote the corresponding homogenized operator. Lemma 3.6 and (3.3) yield

$$\|(\mathcal{H}_*^0 - i)^{-1}\|_{L_2(\Omega) \rightarrow L_2(\theta^\varepsilon)} \leq C\varepsilon,$$

where  $C$  is a constant independent of  $\varepsilon$ . Since  $L_2(\Omega) = L_2(\Omega^\varepsilon) \oplus L_2(\theta^\varepsilon)$ , the latter estimate and (7.2) yield

$$\|(\mathcal{H}^\varepsilon \oplus \mathcal{H}^\theta - i)^{-1} - (\mathcal{H}_*^0 - i)^{-1}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \rightarrow 0, \quad \varepsilon \rightarrow +0.$$

By [37, Ch. VIII, Sec. 7, Ths. VIII.23, VIII.24] it follows the convergence of the spectrum of  $\mathcal{H}^\varepsilon \oplus \mathcal{H}^\theta$  to that of  $\mathcal{H}_*^0$ . And now it remains to employ (7.1) to complete the proof of Theorem 2.5.

## 7.2 Proof of Theorem 2.6

Let

$$f \in L_2(\square^\varepsilon), \quad u^\varepsilon := \left( \mathcal{H}^\varepsilon(\tau) - \frac{\tau^2}{\varepsilon^2} \right)^{-1} f, \quad F(x_2) := \frac{1}{\varepsilon\pi} \int_{-\varepsilon\pi/2}^{\varepsilon\pi/2} f(x) dx_1, \quad (7.3)$$

$$u^0 := (\mathcal{Q}_D^0)^{-1} F, \quad v^\varepsilon := u^\varepsilon - (1 - \chi_1^\varepsilon) u^0, \quad \chi_1^\varepsilon(x) := \chi_1 \left( \frac{x_2 - d_0}{R_2 \varepsilon} \right),$$

where  $R_2$  is chosen so that assumption (A2) holds true. The function  $F$  satisfies the estimate

$$\varepsilon\pi \|F\|_{L_2(0,d)}^2 \leq \|f\|_{L_2(\square)}^2, \quad (7.4)$$

while  $v^\varepsilon$  vanishes on  $\partial\omega^\varepsilon$  and satisfies periodic boundary conditions on  $\partial\square^\varepsilon \setminus \overline{\Gamma}^\varepsilon$ . We extend  $v^\varepsilon$  by zero inside  $\omega^\varepsilon$ .

Proceeding as in (4.1), (4.2), (4.3), (4.4), one can check easily that  $v^\varepsilon$  satisfies the integral identity

$$\begin{aligned} & \left\| \left( i \frac{\partial}{\partial x_1} - \frac{\varsigma}{\varepsilon} \right) v^\varepsilon \right\|_{L_2(\square^\varepsilon)}^2 - \frac{\varsigma^2}{\varepsilon^2} \|v^\varepsilon\|_{L_2(\square^\varepsilon)}^2 + \left\| \frac{\partial v^\varepsilon}{\partial x_2} \right\|_{L_2(\square^\varepsilon)}^2 = (f - F, v^\varepsilon)_{L_2(\square^\varepsilon)} \\ & + ((1 - \chi_1^\varepsilon)F, v^\varepsilon)_{L_2(\square^\varepsilon)} + \left( \frac{\partial \chi_1^\varepsilon}{\partial x_2} \frac{\partial u^0}{\partial x_2}, v^\varepsilon \right)_{L_2(\square^\varepsilon)} - \left( u^0 \frac{\partial \chi_1^\varepsilon}{\partial x_2}, \frac{\partial v^\varepsilon}{\partial x_2} \right)_{L_2(\square^\varepsilon)}. \end{aligned} \quad (7.5)$$

By  $v_\perp^\varepsilon$  we denote the projection of  $v^\varepsilon$  on  $\mathfrak{L}_\perp^\varepsilon$  and see that

$$f - F \in \mathfrak{L}_\perp^\varepsilon, \quad (f - F, v^\varepsilon)_{L_2(\square^\varepsilon)} = (f - F, v_\perp^\varepsilon)_{\square^\varepsilon}.$$

We apply Lemma 4.2 from [10] and (7.4) that implies

$$\begin{aligned} & \left\| \left( i \frac{\partial}{\partial x_1} - \frac{\varsigma}{\varepsilon} \right) v^\varepsilon \right\|_{L_2(\square^\varepsilon)}^2 - \frac{\varsigma^2}{\varepsilon^2} \|v^\varepsilon\|_{L_2(\square^\varepsilon)}^2 \geq \frac{2\varsigma_0}{\varepsilon^2} \left( \|v_\perp^\varepsilon\|_{L_2(\square^\varepsilon)}^2 + \left\| \frac{\partial v_\perp^\varepsilon}{\partial x_1} \right\|_{L_2(\square^\varepsilon)}^2 \right), \quad (7.6) \\ & |(f - F, v^\varepsilon)_{L_2(\square^\varepsilon)}| \leq \|f\|_{L_2(\square^\varepsilon)} \|v_\perp^\varepsilon\|_{L_2(\square^\varepsilon)}. \end{aligned}$$

It follows from (4.5), (4.8), and (3.5) that

$$\begin{aligned} & \|u^0\|_{L_2(\text{supp } \chi_1^\varepsilon)} \leq C\varepsilon^{3/2} \|u^0\|_{W_2^2(\square^\varepsilon)} \leq C\varepsilon^{3/2} \|f\|_{L_2(\square)}, \quad (7.7) \\ & \left\| \frac{\partial u^0}{\partial x_2} \right\|_{L_2(\text{supp } \chi_1^\varepsilon)} \leq C\varepsilon^{1/2} \|f\|_{L_2(\square^\varepsilon)}, \\ & \|v^\varepsilon\|_{L_2(\text{supp } \chi_1^\varepsilon)} \leq C\varepsilon(|\ln \eta|^{1/2} + 1) \|\nabla v^\varepsilon\|_{L_2(\square^\varepsilon)}. \end{aligned}$$

Here and till the end of the proof by  $C$  we indicate inessential constants independent of  $\varepsilon$ ,  $\varsigma_0$ ,  $f$ ,  $u^0$ , and  $v^\varepsilon$ . We substitute these estimates and (7.6) into (7.5) and get

$$\begin{aligned} & \frac{2\varsigma_0}{\varepsilon^2} \|v_\perp^\varepsilon\|_{L_2(\square)}^2 + 4\varsigma_0 \left\| \frac{\partial v^\varepsilon}{\partial x_1} \right\|_{L_2(\square^\varepsilon)}^2 + \left\| \frac{\partial v^\varepsilon}{\partial x_2} \right\|_{L_2(\square^\varepsilon)}^2 \leq \|f\|_{L_2(\square^\varepsilon)} \|v_\perp^\varepsilon\|_{L_2(\square^\varepsilon)} \\ & + C\varepsilon^{1/2} (|\ln \eta|^{1/2} + 1) \|\nabla v^\varepsilon\|_{L_2(\square^\varepsilon)} \|f\|_{L_2(\square^\varepsilon)} \end{aligned}$$

and hence

$$\|\nabla v^\varepsilon\|_{L_2(\square^\varepsilon)} \leq C\varepsilon^{1/2} (|\ln \eta|^{1/2} + 1) \|f\|_{L_2(\square^\varepsilon)}.$$

Since  $v^\varepsilon$  vanishes on  $\Gamma^\varepsilon$ , we have

$$\|v^\varepsilon\|_{L_2(\square^\varepsilon)} \leq C \left\| \frac{\partial v^\varepsilon}{\partial x_2} \right\|_{L_2(\square^\varepsilon)} \leq C\varepsilon^{1/2} (|\ln \eta|^{1/2} + 1) \|f\|_{L_2(\square^\varepsilon)}, \quad (7.8)$$

and together with (7.7) it yields (2.15).

We employ (2.15) to prove (2.16). We follow the main lines of the proof of Theorem 2.5 to show that

$$\left\| \left( \mathcal{H}^\varepsilon(\tau) \oplus \left( \mathcal{H}^\theta + \frac{\tau^2}{\varepsilon^2} \right) - \frac{\tau^2}{\varepsilon^2} \right)^{-1} - (\mathcal{Q}_D^0)^{-1} \oplus 0 \right\|_{L_2(\square^\varepsilon) \rightarrow L_2(\square^\varepsilon)} \leq 2C_0\varsigma_0\varepsilon^{1/2}(|\ln \eta|^{1/2} + 1)$$

for sufficiently small  $\varepsilon$ . Since due to (7.1) the spectrum of  $\mathcal{H}^\theta + \tau^2\varepsilon^{-2}$  is bounded from below by  $C\varepsilon^{-2}\eta^{-2}$ , we get

$$\left| \left( \lambda_n(\tau, \varepsilon) - \frac{\tau^2}{\varepsilon^2} \right)^{-1} - (\Lambda_n^D)^{-1} \right| \leq 2C_0\varsigma_0\varepsilon^{1/2}(|\ln \eta|^{1/2} + 1). \quad (7.9)$$

Given  $N$ , we choose  $\varepsilon_0$  so that the latter estimate holds and  $2C_0\varsigma_0\varepsilon^{1/2}(|\ln \eta|^{1/2} + 1) \leq (2\Lambda_N^D)^{-1}$  for all  $\varepsilon \leq \varepsilon_0$ . In view of ordering of  $\Lambda_n$  it follows from (7.9) that

$$(2\Lambda_n^D)^{-1} \leq \left( \lambda_n(\tau, \varepsilon) - \frac{\tau^2}{\varepsilon^2} \right)^{-1} \leq 3(2\Lambda_n^D)^{-1}, \quad \left| \lambda_n(\tau, \varepsilon) - \frac{\tau^2}{\varepsilon^2} \right| \leq 2\Lambda_n^D.$$

We substitute the latter estimate into (7.9) and arrive at (2.16).

It remains to prove (2.17). In view of (2.14) it sufficient to show that  $\inf_{\tau \in [-1, 1]} \lambda_1(\tau, \varepsilon) = \lambda_1(0, \varepsilon)$ . Let  $\psi^\varepsilon(x)$  be the eigenfunction associated with  $\lambda_1(0, \varepsilon)$  and normalized in  $L_2(\square^\varepsilon \setminus \omega^\varepsilon)$ . The operator  $\mathcal{H}^\varepsilon(\tau)$  commutes with complex conjugation and therefore the eigenfunction  $\psi^\varepsilon$  can be chosen being real-valued. Then it follows from (2.13) and the minimax principle that

$$\lambda_1(\tau, \varepsilon) \leq \mathfrak{h}_\tau^\varepsilon(\psi^\varepsilon, \psi^\varepsilon) = \mathfrak{h}_0^\varepsilon(\psi^\varepsilon, \psi^\varepsilon) + \frac{\tau^2}{\varepsilon^2} = \lambda_1(0, \varepsilon) + \frac{\tau^2}{\varepsilon^2}.$$

This inequality implies the desired identity.

### 7.3 Proof of Theorem 2.7

We define the functions  $f$ ,  $u^\varepsilon$ , and  $F$  by (7.3) and choose  $R_2$  so that assumption (A2) is satisfied. We let  $u^0 := (\mathcal{Q}_K^0(\mu))^{-1}F$ , and by  $\tilde{u}^0$  we denote the solution to the boundary value problem

$$\begin{aligned} -\frac{d^2 \tilde{u}^0}{dx_2^2} &= F \quad \text{in} \quad (0, d) \setminus \{-(R_2 + 1)\varepsilon\}, \quad [\tilde{u}^0]_{x_2=-(R_2+1)\varepsilon} = 0, \\ \left[ \frac{d\tilde{u}^0}{dx_2} \right]_{x_2=-(R_2+1)\varepsilon} - 2(K + \mu)\tilde{u}^0|_{x_2=-(R_2+1)\varepsilon} &= 0. \end{aligned} \quad (7.10)$$

We define an analogue of function (6.6),

$$W^\varepsilon(x) := \begin{cases} -\frac{1}{\ln \eta(\varepsilon)} \ln \frac{|x|}{R_2\varepsilon\eta(\varepsilon)}, & x \in B_{R_2\varepsilon}(y_0^\varepsilon) \setminus B_{R_2\varepsilon\eta(\varepsilon)}(y_0^\varepsilon), \\ 0, & x \in B_{R_2\varepsilon\eta(\varepsilon)}(y_0^\varepsilon), \\ 1, & x \in \square^\varepsilon \setminus B_{R_2\varepsilon}(y_0^\varepsilon). \end{cases}$$

Up to obvious minor changes, this function has the same properties as function (6.6). We define  $v^\varepsilon := u^\varepsilon - W^\varepsilon\tilde{u}^0$ , extend this function by zero inside  $\omega^\varepsilon$  and write the boundary value problem for  $v^\varepsilon$ ,

$$\left( \left( i \frac{\partial}{\partial x_1} - \frac{\varsigma}{\varepsilon} \right)^2 - \frac{\partial^2}{\partial x_2^2} - \frac{\varsigma^2}{\varepsilon^2} \right) v^\varepsilon = f_1 \quad \text{in} \quad \square^\varepsilon \setminus (\omega^\varepsilon \cup \partial B_{R_2\varepsilon}(y_0^\varepsilon) \cup B_{R_2\varepsilon\eta}(y_0^\varepsilon)),$$

$$\begin{aligned}
f_1 &:= f - W^\varepsilon F + 2i \frac{\varsigma}{\varepsilon} \frac{\partial W^\varepsilon}{\partial x_1} \tilde{u}^0 + 2 \frac{\partial W^\varepsilon}{\partial x_2} \frac{d\tilde{u}^0}{dx_2}, \quad v^\varepsilon = 0 \quad \text{on} \quad \Gamma^\varepsilon \cup \partial\omega^\varepsilon, \\
[v^\varepsilon]_{\partial B_{R_2\varepsilon\eta}(y_0^\varepsilon)} &= 0, \quad \left[ \frac{\partial v^\varepsilon}{\partial |x - y_0^\varepsilon|} \right]_{\partial B_{R_2\varepsilon\eta}(y_0^\varepsilon)} = \frac{\tilde{u}^0|_{\partial B_{R_2\varepsilon\eta}(y_0^\varepsilon)}}{R_2\varepsilon\eta \ln \eta}, \\
[v^\varepsilon]_{\partial B_{R_2\varepsilon}(y_0^\varepsilon)} &= 0, \quad \left[ \frac{\partial v^\varepsilon}{\partial |x - y_0^\varepsilon|} \right]_{\partial B_{R_2\varepsilon}(y_0^\varepsilon)} = -\frac{\tilde{u}^0|_{\partial B_{R_2\varepsilon}(y_0^\varepsilon)}}{R_2\varepsilon \ln \eta}, \\
[v^\varepsilon]_{x_2=-(R_2+1)\varepsilon} &= 0, \quad \left[ \frac{\partial v^\varepsilon}{\partial x_2} \right]_{x_2=-(R_2+1)\varepsilon} = 2(K + \mu)\tilde{u}^0|_{x_2=-(R_2+1)\varepsilon},
\end{aligned}$$

and subject to periodic condition on the lateral boundaries of  $\square^\varepsilon$ . We write the associated integral identity employing that  $v^\varepsilon = 0$  in  $\omega^\varepsilon$ ,

$$\begin{aligned}
&\left\| \left( i \frac{\partial}{\partial x_1} - \frac{\varsigma}{\varepsilon} \right) v^\varepsilon \right\|_{L_2(\square^\varepsilon)}^2 + \left\| \frac{\partial v^\varepsilon}{\partial x_2} \right\|_{L_2(\square^\varepsilon)}^2 - \frac{\varsigma^2}{\varepsilon^2} \|v^\varepsilon\|_{L_2(\square^\varepsilon)}^2 = (f_1, v^\varepsilon)_{L_2(\square^\varepsilon)} \\
&+ \frac{1}{R_2\varepsilon \ln \eta} (\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{R_2\varepsilon}(y_0^\varepsilon))} - \frac{1}{R_2\varepsilon \eta \ln \eta} (\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{R_2\varepsilon\eta}(y_0^\varepsilon))} \\
&- 2(K + \mu) (\tilde{u}^0, v^\varepsilon)_{L_2(\{x: x_2=-(R_2+1)\varepsilon\} \cap \square^\varepsilon)}. \tag{7.11}
\end{aligned}$$

Proceeding as in the proof of Lemma 3.2, we obtain

$$\|v^\varepsilon\|_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon))} \leq C\varepsilon^{1/2} \|v^\varepsilon\|_{W_2^1(\square^\varepsilon)}. \tag{7.12}$$

Here and till the end of the proof by  $C$  we indicate inessential constants independent of  $\varepsilon$ ,  $\varsigma_0$ ,  $f$ ,  $u^0$ , and  $v^\varepsilon$ . We employ this estimate, (7.4), Lemma 4.2 in [10] and the identity  $f - W^\varepsilon F = f - F + (1 - W^\varepsilon)F$  to obtain

$$((f - W^\varepsilon F), v^\varepsilon)_{L_2(\square^\varepsilon)} \leq \|f\|_{L_2(\square)} \|v_\perp^\varepsilon\|_{L_2(\square)} + C\varepsilon \|f\|_{L_2(\square^\varepsilon)} \|v^\varepsilon\|_{W_2^1(\square^\varepsilon)}, \tag{7.13}$$

where  $v_\perp^\varepsilon$  is the projection of  $v^\varepsilon$  on  $\mathfrak{L}_\perp^\varepsilon$ .

Problem (7.10) can be solved explicitly and its solution obeys the estimate

$$\|\tilde{u}^0\|_{C[0,d]} + \left| \frac{d\tilde{u}^0}{dx_2} \right|_{C[0,d]} \leq C\|F\|_{L_2(0,d)} \leq C\varepsilon^{-1/2} \|f\|_{L_2(\square^\varepsilon)}, \tag{7.14}$$

where we have also used (7.4). We integrate by parts as follows,

$$\begin{aligned}
2i \frac{\varsigma}{\varepsilon} \left( \frac{\partial W^\varepsilon}{\partial x_1} \tilde{u}^0, v^\varepsilon \right)_{L_2(\square^\varepsilon)} &= 2i \frac{\varsigma}{\varepsilon} \left( \frac{\partial(W^\varepsilon - 1)}{\partial x_1} \tilde{u}^0, v^\varepsilon \right)_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_0^\varepsilon))} \\
&= 2i \frac{\varsigma}{\varepsilon} \left( \tilde{u}^0 \frac{\partial x_1}{\partial |x - y_0^\varepsilon|}, v^\varepsilon \right)_{L_2(\partial B_{R_2\varepsilon}(y_0^\varepsilon))} + 2i \frac{\varsigma}{\varepsilon} \left( (1 - W^\varepsilon) \tilde{u}^0, \frac{\partial v^\varepsilon}{\partial x_1} \right)_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_0^\varepsilon))}.
\end{aligned}$$

These identities, Lemma 6.2, and (7.14) yield

$$\begin{aligned}
\left| 2i \frac{\varsigma}{\varepsilon} \left( \frac{\partial W^\varepsilon}{\partial x_1} \tilde{u}^0, v^\varepsilon \right)_{L_2(\square^\varepsilon)} \right| &\leq C\varepsilon^{-1} \|\tilde{u}^0\|_{L_2(\partial B_{R_2\varepsilon\eta}(y_0^\varepsilon))} \|v^\varepsilon\|_{L_2(\partial B_{R_2\varepsilon\eta}(y_0^\varepsilon))} \\
&+ C\varepsilon^{-1} \|\tilde{u}^0\|_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon))} \left\| \frac{\partial u^\varepsilon}{\partial x_1} \right\|_{L_2(\square^\varepsilon)} \\
&\leq C\varepsilon^{-1/2} \|f\|_{L_2(\square^\varepsilon)} \|\nabla v^\varepsilon\|_{L_2(\square^\varepsilon)} + C\varepsilon^{-1/2} \|f\|_{L_2(\square^\varepsilon)} \left\| \frac{\partial v^\varepsilon}{\partial x_1} \right\|_{L_2(\square^\varepsilon)}. \tag{7.15}
\end{aligned}$$

In the same way we have

$$\begin{aligned}
& \left| 2 \left( \frac{\partial W^\varepsilon}{\partial x_2} \frac{d\tilde{u}^0}{dx_2}, v^\varepsilon \right)_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_0^\varepsilon))} \right| \\
& \leq C \|f\|_{L_2(\square)} \left\| \frac{\partial W^\varepsilon}{\partial x_2} \right\|_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon) \setminus B_{R_2\varepsilon\eta}(y_0^\varepsilon))} \|v^\varepsilon\|_{W_2^1(\square^\varepsilon)} \\
& \leq \frac{C}{|\ln \eta|^{1/2}} \|f\|_{L_2(\square^\varepsilon)} \|v^\varepsilon\|_{W_2^1(\square^\varepsilon)}, \\
& \left| \frac{(\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{R_2\varepsilon\eta}(y_0^\varepsilon))}}{R_2\varepsilon \ln \eta} \right| \leq \frac{C}{|\ln \eta|} \varepsilon^{-1/2} \|f\|_{L_2(\square^\varepsilon)} \|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}.
\end{aligned} \tag{7.16}$$

Proceeding as in (5.20), (5.18), one can easily make sure that

$$\begin{aligned}
& \left| \frac{(\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{R_2\varepsilon}(y_0^\varepsilon))}}{R_2\varepsilon \ln \eta} - 2(K + \mu)(\tilde{u}^0, v^\varepsilon)_{L_2(\{x: x_2 = -(R_2+1)\varepsilon\} \cap \square^\varepsilon)} \right| \\
& \leq C(K + \mu) \left( \|\tilde{u}^0\|_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon))} \|\nabla v^\varepsilon\|_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon))} + \left\| \frac{d\tilde{u}^0}{dx_2} \right\|_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon))} \|v^\varepsilon\|_{L_2(B_{R_2\varepsilon}(y_0^\varepsilon))} \right),
\end{aligned}$$

and by (7.14), (7.4), (7.12) it implies

$$\begin{aligned}
& \left| \frac{(\tilde{u}^0, v^\varepsilon)_{L_2(\partial B_{R_2\varepsilon}(y_0^\varepsilon))}}{R_2\varepsilon \ln \eta} - 2(K + \mu)(\tilde{u}^0, v^\varepsilon)_{L_2(\{x: x_2 = -(R_2+1)\varepsilon\} \cap \square^\varepsilon)} \right| \\
& \leq C\varepsilon^{1/2}(K + \mu) \|f\|_{L_2(\square^\varepsilon)} \|v^\varepsilon\|_{L_2(\square^\varepsilon)}.
\end{aligned}$$

This estimate and (7.16), (7.15), (7.13), (7.11), (7.6), (6.3) lead us to the inequality

$$\begin{aligned}
& \frac{2\zeta^0}{\varepsilon^2} \left( \|v_\perp^\varepsilon\|_{L_2(\square^\varepsilon)}^2 + \left\| \frac{\partial v^\varepsilon}{\partial x_1} \right\|_{L_2(\square^\varepsilon)}^2 \right) + \left\| \frac{\partial v^\varepsilon}{\partial x_2} \right\|_{\square^\varepsilon}^2 \leq \|f\|_{L_2(\square^\varepsilon)} \|v_\perp^\varepsilon\|_{L_2(\square^\varepsilon)} \\
& + C\varepsilon^{-1/2} \|f\|_{L_2(\square^\varepsilon)} \left\| \frac{\partial v^\varepsilon}{\partial x_1} \right\|_{L_2(\square^\varepsilon)} + C(\varepsilon + \varepsilon^{1/2}(K + \mu)^{1/2}) \|f\|_{L_2(\square^\varepsilon)} \|v^\varepsilon\|_{W_2^1(\square^\varepsilon)}.
\end{aligned}$$

Hence, by the first inequality in (7.8),

$$\|v^\varepsilon\|_{L_2(\square^\varepsilon)} \leq C\varepsilon^{1/2} \|f\|_{L_2(\square^\varepsilon)}.$$

Extending  $u^\varepsilon$  by zero inside  $\omega^\varepsilon$ , by (7.14) we finally have

$$\begin{aligned}
\|u^\varepsilon - u^0\|_{L_2(\square^\varepsilon)} & \leq \|\tilde{u}^0 - u^0\|_{L_2(\square^\varepsilon)} + \|(1 - W^\varepsilon)\tilde{u}^0\|_{L_2(\square^\varepsilon)} + \|v^\varepsilon\|_{L_2(\square^\varepsilon)} \\
& \leq C\varepsilon^{1/2} \|f\|_{L_2(\square^\varepsilon)}
\end{aligned}$$

that proves (2.18). The rest of the proof is completely the same as in the proof of Theorem 2.6.

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